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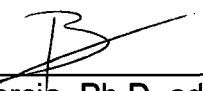
A Model of Strategy Games Based on the Paradigm of the Iterated Prisoner's Dilemma Employing Fuzzy Sets

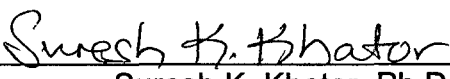
Esta tese foi julgada adequada para a obtenção título de "Doutor em Engenharia", especialidade Engenharia de Produção e aprovada em sua forma final pelo Programa de Pós-graduação.


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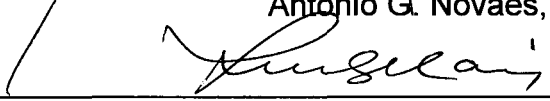

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**A model of strategy games based on the paradigm of the Iterated
Prisoner's Dilemma employing Fuzzy Sets**

**Doctoral Dissertation submitted as a partial fulfillment of the conditions required for
the obtainance of the degree of Doctor in Production and Systems Engineering**

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**Um modelo de jogos de estratégia com base no paradigma do
Dilema do Prisioneiro Iterado, com emprego de Fuzzy Sets**

**Tese submetida como requisito parcial para obtenção do título de
Doutor em Engenharia de Produção**

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Resumo

A Teoria de Jogos pode ser definida como a análise matemática formal de situações onde se observa algum tipo de *conflito de interesses*. Os jogadores são assumidos, pela teoria clássica, como *racionais*, o que significa que buscam sempre no jogo a maximização de suas utilidades esperadas. O jogo de 2 participantes conhecido por "Dilema do Prisioneiro" (PD) é considerado como o que melhor representa a contradição entre o emprego da racionalidade individual e da coletiva. Na versão tradicional do PD, as estratégias são sempre compostas pelas ações elementares "Cooperate" (C) ou "Defect" (D). O jogo iterado (IPD) constitui-se em um modelo que traduz com bastante fidelidade, e de forma simples, inúmeras questões freqüentemente encontradas na realidade, quando pessoas, grupos ou empresas disputam parcelas de recursos escassos ou limitados. A literatura disponível sobre o tema é muito vasta, e muitas análises do IPD, incluindo simulações através de torneios computacionais e uso de técnicas de inteligência artificial já foram efetuadas. Contudo, as estratégias consideradas no jogo estão restritas, na grande maioria dos estudos, a combinações determinísticas, condicionais ou probabilísticas das ações pontuais citadas, ou seja C e D. Nesta tese, uma nova versão do IPD é desenvolvida, chamada de *Fuzzy Iterated Prisoner's Dilemma* — FIPD. Este método, ao permitir que os jogadores possam implementar ações graduais, objetiva uma modelagem mais realista do conflito de interesses representado pelo jogo. Adicionalmente, no FIPD os agentes decidem com base em um raciocínio qualitativo, traduzido neste trabalho pelo uso de *sistemas especialistas difusos*. A tese inclui dois enfoques do FIPD. O primeiro, de caráter exploratório, consiste em um torneio computacional confrontando os estrategistas assistidos pelo sistema de decisão difuso entre si e com outros tipos de jogadores tradicionalmente bem sucedidos em trabalhos anteriores (TFT e *Pavlov*). O segundo enfoque, bem mais elaborado e detalhado, trata de uma aplicação prática a um problema de divisão de mercado, onde várias firmas atuam vendendo produtos ou serviços a uma população diversificada de compradores, que possuem diferentes poder de compra e preferências relativas à qualidade dos itens. Esta abordagem imprimiu uma característica altamente dinâmica às interações, simuladas com auxílio de um programa computacional orientado a objetos especialmente desenvolvido. As estratégias empregadas pelos participantes e seus desempenhos são analisados e discutidos, especialmente na segunda abordagem do FIPD, obtendo-se importantes conclusões a respeito do problema.

Abstract

Game Theory can be defined as the formal mathematical analysis of situations where some kind of *conflict of interest* exists. The classical theory assumes that the players are rational, in the sense that they always seek the maximization of their expected utilities. The 2-person game known by the name of "Prisoner's Dilemma" (PD) is considered as the situation which best depicts the contradiction between the individual and the collective rationality. In its traditional version, the strategies in the PD are always composed by the elementary and dichotomic actions "Cooperate" (C) ou "Defect" (D). The iterated game (IPD) constitutes a model that mirrors, in a simple and direct manner, several problems often found in the real world, when people, groups or companies are involved in a dispute over some limited or scarce resource. The available literature regarding that theme is vast, and many comprehensive analysis of the IPD have been already performed, including computational tournaments and the employment of artificial intelligence techniques. However, the strategies considered in the game are usually confined to deterministic or probabilistic combinations of the mentioned punctual actions C and D. In this dissertation, an innovative version of the IPD is presented and developed, called *Fuzzy Iterated Prisoner's Dilemma* — FIPD. That method, by allowing the players to implement *gradual* actions, aims at attaining a more realistic model of the conflict of interested represented by the game. Additionally, in the FIPD the agents decide based on a qualitative system of reasoning, which in this work regards the utilization of *fuzzy expert systems* (FES). The dissertation includes two approaches of the FIPD. The first has an exploratory character, and consists of a computational contest where the FES-assisted strategists confront each other and also other well known successful rules (TFT and *Pavlov*). The second approach, much more elaborate and perfected, is directed at a practical application concerning a market share problem. There, several firms are present offering their products or services to a diversified population of buyers, the Consumers, which on their turn possess different buying powers and preferences relative to the quality of itens desired. That method has the advantage of imprinting a highly dynamic feature to the environment, which has been simulated by means of an object-oriented C++ program especially written. The strategies employed by the participants and their respective performances are extensively analyzed and discussed, mainly in the second approach, from which many important conclusions have been drawn.

Contents

| | Page |
|---|------|
| Resumo | v |
| Abstract | vi |
| Chapter 1 | |
| Introduction | |
| 1.1 Preamble | 1.1 |
| 1.2 The Prisoner's Dilemma | 1.2 |
| 1.3 Description of the Problem and Objective of the Dissertation | 1.3 |
| 1.4 Methodology | 1.6 |
| 1.5 Structure and Outline of the Dissertation | 1.8 |
| 1.6 Main Contributions of the Dissertation | 1.10 |
| Chapter 2 | |
| A Review of Fundamental Topics of Game Theory | |
| 2.1 Introduction | 2.1 |
| 2.2 Representation of Strategic Games | 2.4 |
| 2.2.1 Decision Trees | 2.4 |
| 2.2.2 Rule Sets | 2.5 |
| 2.3 Usual Methods for the Evaluation of the Sub-optimal Function | 2.6 |
| 2.3.1 The Minimax Method | 2.6 |
| 2.3.2 Alpha-Beta Cuts | 2.7 |
| 2.3.3 Other Methods | 2.8 |

| | |
|---|------|
| 2.4 Information Sets | 2.8 |
| 2.5 Strategies | 2.9 |
| 2.5.1 Saddle Points | 2.10 |
| 2.5.2 Pure and Mixed Strategies | 2.11 |
| 2.5.3 The Minimax Theorem | 2.14 |
| 2.6 Preferences, Utility and Rationality | 2.16 |
| 2.6.1 Fundamental Axioms of Utility Theory | 2.17 |
| 2.6.2 Prospect Theory | 2.19 |
| 2.6.3 Rationality and the Maximization of Expected Utility | 2.22 |
| 2.6.4 Other Criticisms to the Maximum Expected Utility Method | 2.25 |
| 2.7 Extensive and Normal Forms of a Game | 2.26 |
| 2.7.1 Extensive Form | 2.26 |
| 2.7.2 Normal Form | 2.27 |
| 2.8 Competitive Non-Zero-Sum Games | 2.27 |
| 2.9 Dominant and Nash Equilibria | 2.28 |
| 2.9.1 Dominant Strategy Equilibrium | 2.29 |
| 2.9.2 Nash Equilibrium | 2.30 |
| 2.10 Cooperative Games | 2.32 |
| 2.10.1 Two-Person Cooperative Games | 2.33 |
| 2.10.2 Von Neumann and Morgenstern Solution | 2.34 |
| 2.10.3 Arbitration Schema | 2.39 |
| 2.10.4 Nash's Arbitration Scheme | 2.40 |
| 2.10.5 Criticisms to Nash's Scheme | 2.42 |
| 2.10.6 Other Arbitration Models | 2.42 |
| 2.11 Cournot and Bertrand Games | 2.44 |
| 2.11.1 Cournot's Theory | 2.44 |
| 2.11.2 Bertrand's Model | 2.46 |
| 2.11.3 Some Variations on the Bertrand Game | 2.48 |
| 2.11.4 Concluding Remarks | 2.51 |

Chapter 3

The Prisoner's Dilemma Game: A Survey

| | |
|---|------|
| 3.1 Introduction | 3.1 |
| 3.2 The Prisoner's Dilemma in the Context of 2×2 Games | 3.5 |
| 3.2.1 Some Peculiar Social Dilemmas | 3.5 |
| 3.2.2 The Effect of Preferences in 2×2 Games | 3.7 |
| 3.2.3 Other Related 2×2 Games | 3.11 |
| 3.3 Computational Tournaments of the Prisoner's Dilemma | 3.11 |
| 3.3.1 Overview | 3.11 |
| 3.3.2 Aspects of the Tournaments' Dynamics | 3.12 |
| 3.4 Evolutionary Games, Spatial and AI Techniques Approaches of the PD | 3.19 |
| 3.4.1 An Evolutionary Biological Application: The Hawk-Dove Game | 3.19 |
| 3.4.2 Spatial Models of the IPD | 3.24 |
| 3.4.3 Singular Reactive Strategies | 3.25 |
| 3.4.4 The IPD with AI-aided Players | 3.27 |
| 3.5 The One-sided Prisoner's Dilemma | 3.39 |
| 3.6 Applications of PD-related Games to Economic Problems ... | 3.41 |
| 3.6.1 International Trade Tariff Policy | 3.41 |
| 3.6.2 Share Takeover | 3.42 |
| 3.6.3 An Oligopoly Game | 3.43 |
| 3.7 Concluding Remarks | 3.45 |

Chapter 4

A Review of Fuzzy Set Theory and Expert Systems

| | |
|---|-----|
| 4.1 Introduction | 4.1 |
| 4.2 Fuzzy Measures and Fuzzy Set Theory | 4.2 |
| 4.2.1 General Discussion | 4.2 |
| 4.2.2 Fuzzy Measures | 4.4 |

| | |
|--|------|
| 4.2.3 Belief and Plausibility Measures | 4.5 |
| 4.2.4 Fuzzy Set Theory | 4.10 |
| 4.3 Operations with Fuzzy Sets | 4.13 |
| 4.3.1 Basic Concepts | 4.13 |
| 4.3.2 The Extension Principle | 4.16 |
| 4.3.3 Triangular Norms and Co-norms | 4.17 |
| 4.3.4 The Fuzzy Integral | 4.18 |
| 4.3.5 Modification of the Fuzzy Integral | 4.21 |
| 4.3.6 Linguistic Hedges | 4.24 |
| 4.4 Fuzzy Expert Systems | 4.27 |
| 4.4.1 General Discussion | 4.27 |
| 4.4.2 Fuzzy Logic and Fuzzy Inference | 4.29 |
| 4.4.3 Defuzzification Methods | 4.32 |

Chapter 5

A Fuzzy Approach to the Prisoner's Dilemma

| | |
|--|------|
| 5.1 Introduction | 5.1 |
| 5.2 A Payoff Function | 5.3 |
| 5.3 Fuzzy Decision Rules | 5.6 |
| 5.3.1 Relation Between Accumulated Wealths | 5.6 |
| 5.3.2 Last Iterations between the Parties | 5.9 |
| 5.3.3 Relation between overall Trends of Wealth | 5.11 |
| 5.4 Determination of a Player's Action in a Move | 5.12 |
| 5.5 Example of an Iteration of the FIPD | 5.13 |
| 5.5.1 Identification of the Players | 5.13 |
| 5.5.2 The Iteration Process | 5.13 |
| 5.5.3 The Decision Process for a Player | 5.14 |
| 5.5.4 Computation of the Payoffs | 5.16 |
| 5.5.5 Update of the Population and Players' Parameters | 5.16 |
| 5.6 Simulations | 5.17 |

| | |
|-------------------------------------|------|
| 5.7 Discussion of the Results | 5.19 |
| 5.8 Conclusions | 5.20 |

Chapter 6

An Application of the Fuzzy Iterated Prisoner's Dilemma

| | |
|---|------|
| 6.1 Introduction | 6.1 |
| 6.2 Description of the Problem | 6.2 |
| 6.3 Methodology Overview | 6.4 |
| 6.3.1 The Basic Game | 6.4 |
| 6.3.2 Basic Iteration Process | 6.8 |
| 6.4 Detailed Formulation of the One-sided Fuzzy IPD Market Share Game | 6.10 |
| 6.4.1 The Sellers | 6.10 |
| 6.4.2 The Advertising Budget | 6.11 |
| 6.4.3 The Size of the Potential Market | 6.12 |
| 6.4.4 The Firms' Revenues, Costs and Profit | 6.13 |
| 6.4.5 The Consumers | 6.19 |
| 6.4.6 The Evaluation of a Product or Service by the Consumer | 6.37 |
| 6.4.7 The Consumers' Payoffs | 6.45 |
| 6.4.8 The Consumer's Decision | 6.55 |
| 6.5 Summary of the Market Share Game | 6.67 |

Chapter 7

Simulation of the One-sided Fuzzy IPD Market Share Game

| | |
|--|-----|
| 7.1 The Simulation Program | 7.1 |
| 7.2 Methodology of the Experiments | 7.3 |
| 7.2.1 Bookkeeping of the Results | 7.3 |
| 7.2.2 Scope of the Simulations | 7.4 |
| 7.3 Analysis and Discussion of the Results | 7.8 |
| 7.3.1 General Remarks | 7.8 |

| | |
|--|------|
| 7.3.2 The Simulations of Phase I | 7.10 |
| 7.3.3 The Simulations of Phase II | 7.22 |
| 7.3.4 The Simulations of Phase III | 7.35 |
| 7.3.5 The Simulations of Phase IV | 7.40 |
| 7.4 Final Remarks | 7.49 |

Chapter 8

Conclusions

| | |
|--|-----|
| 8.1 Summary of the Dissertation | 8.1 |
| 8.2 Synopsis of the Results | 8.2 |
| 8.2.1 The FIPD Tournaments | 8.2 |
| 8.2.2 The 1S-FIPD and the Market Share Game | 8.4 |
| 8.3 Main Contributions of the Work | 8.6 |
| 8.4 Limitations of the Work and Suggestions for Future Developments | 8.7 |
| 8.4 Final Remarks | 8.9 |

Appendix

| | |
|---|----------|
| App. A Listing of the Consumers' Frequency and Incomes | App. 6.a |
| App. B Graphs of the Attractiveness to Price and Quality Functions | App. 6.b |

| | |
|-----------------------------------|-------|
| References and Bibliography | Ref.1 |
|-----------------------------------|-------|

Figures

| | |
|---|------|
| 2.1 Fragment of a Decision Tree with the Node's Gains attributed by the Minimax Method | 2.6 |
| 2.2 Payoff Matrix of a Non-strictly Competitive Game I | 2.28 |
| 2.3 Payoff Matrix of a Non-strictly Competitive Game II | 2.31 |
| 2.4 Payoff Matrix of a Strictly Solvable Competitive Game | 2.32 |
| 2.5 Payoff Matrix of a Cooperative Game | 2.34 |

| | |
|--|------|
| 2.6 Negotiation Set R for the Von Neumann and Morgenstern Solution of a Cooperative Game with Two Independent Mixed Strategies | 2.35 |
| 2.7 Cooperative Solution Set R' and Pareto's Optimal Set | 2.37 |
| 2.8 Pareto's Optimal Set and the Negotiation Set | 2.38 |
| 2.9 Space of Feasible Agreements with the <i>status quo</i> point | 2.41 |
| 2.10 Reaction Functions in the Cournot Duopoly Game | 2.46 |
| 2.11 Bertrand Reaction Functions and Equilibrium Price with Differentiated Products | 2.50 |
| | |
| 3.1 Generic Payoff Matrix of the Prisoner's Dilemma | 3.3 |
| 3.2 A FSM that plays TFT in the IPD | 3.29 |
| 3.3 Schematic Diagram of a Best Evolved FSM with Seven States | 3.36 |
| 3.4 A Multi-layer Perceptron for the IPD with continuous Actions and Payoffs | 3.37 |
| 3.5 Normal Form of the Product Quality Game - An One-sided PD | 3.40 |
| 3.6 Payoff Matrix of a Cournot Duopolistic Game | 3.44 |
| | |
| 4.1 Relationship among the Main Classes of Fuzzy Measures | 4.4 |
| 4.2 Carnival Wheel with a Hidden Sector | 4.7 |
| 4.3 Some Shapes Commonly employed for the Membership Function $\mu_S(x)$ representing a Fuzzy Set S | 4.11 |
| 4.4 Comparison of Min, Max and Bounded Sum Operators | 4.14 |
| 4.5 Example of Fuzzy Sets that result from the Application of Linguistic Hedges | 4.25 |
| 4.6 Example of Three Criteria for determining the Centroid using the Center of Gravity Method | 4.34 |
| | |
| 5.1 Fuzzy Sets describing Possible Actions in the FIPD | 5.4 |
| 5.2 The FIPD's Payoff Function | 5.5 |
| 5.3 Fuzzy Sets for the Qualitative Description of f_1 | 5.7 |
| 5.4 Fuzzy Sets for the Qualitative Description of f_2 | 5.10 |
| | |
| 6.1 Normal Form of the Simplified PD with Qualitative Payoffs | 6.6 |
| 6.2 Relation between the Cost q_{ig} and the cost c_{ig} | 6.18 |
| 6.3 Consumers' Income Frequency Histogram | 6.22 |

| | |
|--|------|
| 6.4 Fuzzy Sets describing the Consumers' Normalized Income | 6.24 |
| 6.5 Fuzzy Sets describing the Consumers' Sensitivity to Price and Quality | 6.27 |
| 6.6 Basic Fuzzy Sets describing the price p_{ig} and the quality q_{ig} | 6.31 |
| 6.7 Example of the Defuzzification of The Sensitivity to Price Parameter | 6.35 |
| 6.8 Modified Fuzzy Sets according to a Consumer's Sensitivity to Price | 6.36 |
| 6.9 Modified Fuzzy Sets according to a Consumer's Sensitivity to Quality | 6.37 |
| 6.10 Fuzzy Sets depicting the Attractiveness of the price p_{ig} | 6.39 |
| 6.11 Fuzzy Sets depicting the Attractiveness of the quality q_{ig} | 6.40 |
| 6.12 The Importance Factors of the Attributes Price, Quality | 6.43 |
| 6.13 Attractiveness Indexes α_{pj} and α_{qj} | 6.47 |
| 6.14 Range of the Synthetic Evaluation as a Function of the Consumers' Sensitivities to Price | 6.48 |
| 6.15 Some Estimated Payoffs Functions | 6.59 |
| | |
| 7.1 Basic Diagram of the Simulation Process | 7.2 |
| 7.2 Syenthetic Evaluation Ranges & Optimum Prices | 7.10 |
| 7.3(a-e) Ratio of Success — Game 1, Cycle 1 | 7.11 |
| 7.4 Cycle 1: % of Total Successful Iterations with All Firms | 7.13 |
| 7.5 Cycle 1— Market Share | 7.13 |
| 7.6 Cycle 1— Profit | 7.13 |
| 7.7 Cycle 1— Comparison between Profit and Market Share | 7.14 |
| 7.8 Evolution of the Success Ratio for Consumers of Class 2 | 7.15 |
| 7.9 Evolution of the Success Ratio for Consumers of Class 3 | 7.15 |
| 7.10 Evolution of the Success Ratio for Consumers of Class 4 | 7.16 |
| 7.11 Evolution of the Success Ratio for Consumers of Class 5 | 7.16 |
| 7.12 Evolution of the Success Ratio for Consumers of Class 6 | 7.17 |
| 7.13 Evolution of the Success Ratio for Consumers of Class 7 | 7.17 |
| 7.14 Evolution of the Success Ratio for the Population of Consumers ... | 7.18 |
| 7.15 Comparison of Global Profit and Market Share for Firms 0 and 1 ... | 7.19 |
| 7.16 Comparison of Global Profit and Market Share for Firms 2 and 3 ... | 7.19 |
| 7.17 Comparison of Global Profit and Market Share for Firms 4 and 5 ... | 7.20 |

| | |
|---|------|
| 7.18 Firm 0: Success Ratio & Profit as a Function of Cost | 7.23 |
| 7.19 Firm 1: Success Ratio & Profit as a Function of Cost | 7.24 |
| 7.20 Firm 2: Success Ratio & Profit as a Function of Cost | 7.25 |
| 7.21(a) Firm 0: Success Ratio | 7.30 |
| 7.21(b) Firm 0: Profit | 7.30 |
| 7.22(a) Firm 1: Success Ratio | 7.30 |
| 7.22(b) Firm 1: Profit | 7.30 |
| 7.23(a) Firm 2: Success Ratio | 7.30 |
| 7.23(b) Firm 2: Profit | 7.30 |
| 7.24(a) Firm 3: Success Ratio | 7.31 |
| 7.24(b) Firm 3: Profit | 7.31 |
| 7.25(a) Firm 4: Success Ratio | 7.31 |
| 7.25(b) Firm 4: Profit | 7.31 |
| 7.26(a) Firm 5: Success Ratio | 7.31 |
| 7.26(b) Firm 5: Profit | 7.31 |
| 7.27 Evolution of Success for Games 1-4 | 7.32 |
| 7.28 Evolution of Profit for Games 1-4 | 7.32 |
| 7.29 Units sold per Game | 7.33 |
| 7.30 Firms' Profits per Game | 7.33 |
| 7.31 Distribution of Buyers—Game 1 | 7.35 |
| 7.32 Contribution to Firms' Profits from the Consumers' Classes | 7.35 |
| 7.33 Distribution of Successful Iterations | 7.36 |
| 7.34 Distribution of Profits | 7.36 |
| 7.35 Game 1—Successes and Profit as functions of the Markup | 7.37 |
| 7.36 Game 2—Successes and Profit as functions of the Markup | 7.37 |
| 7.37 Game 3—Successes and Profit as functions of the Markup | 7.37 |
| 7.38 Game 4—Successes and Profit as functions of the Markup | 7.37 |
| 7.39 Game 5—Successes and Profit as functions of the Markup | 7.38 |
| 7.40 Game 6—Successes and Profit as functions of the Markup | 7.38 |
| 7.41 Conjoint Success Ratio for Classes 1 & 2 — Game 1 | 7.45 |
| 7.42 Firms' Accumulated Payoffs with Class 1 | 7.46 |
| 7.43 Total Profit in Game 1 (Without Advertising Expenses) | 7.47 |

| | |
|--|------|
| 7.44 Total Profit in Game 2 (Without Advertising Expenses) | 7.47 |
| 7.45 Number of Iterations and Successes— Equivalent Advertising | 7.48 |
| 7.46 Number of Iterations and Successes— Differentiated Advertising .. | 7.48 |

Tables

| | |
|---|------|
| 2.1 Information Categories of a Game | 2.8 |
| 2.2 Allais' Situation 1 | 2.24 |
| 2.3 Allais' Situation 2 | 2.24 |
| 3.1 Four Distinctive 2×2 Dilemmas | 3.6 |
| 3.2 Types of Players and their Ranking of Preferences | 3.9 |
| 3.3 Possible Pairings of Players and the Resulting Joint Elective Strategies | 3.10 |
| 3.4 Payoff Matrix of Axelrod's Computational Tournaments of the IPD .. | 3.13 |
| 3.5 The Hawk-Dove Game's Payoff Matrix | 3.20 |
| 4.1 Mass Function and Belief Intervals for the Carnival Wheel | 4.8 |
| 4.2 A Sample of Linguistic Hedges | 4.26 |
| 4.3 A Boolean Truth Table for the Statements P, Q | 4.30 |
| 5.1 Usual Payoff Matrix of the PD Game | 5.3 |
| 5.2 Fuzzy Production Rules involving the Wealth Relation f_1 and an Action a_1 | 5.8 |
| 5.3 Strategies used by the 512 Different Fuzzy Players of the FIPD | 5.13 |
| 5.4(a) First Phase: Groups 1 to 16 | 5.18 |
| 5.4(b) First Phase: Groups 17 to 32 | 5.18 |
| 5.5 Second Phase | 5.18 |
| 5.6 Third Phase | 5.18 |
| 5.7 Fourth Phase (Final) | 5.19 |
| 6.1 Actions and Payoffs in a Simplified One-Sided Market Share PD .. | 6.8 |
| 6.2 Main Components of the Market Share Game | 6.9 |

| | |
|---|------|
| 6.3 The Firms' Payoffs | 6.14 |
| 6.4 Relative Frequency Distribution of The Consumers' Income | 6.21 |
| 6.5 Specification of the Consumers' Income Fuzzy Sets | 6.23 |
| 6.6 Rules used by the Fuzzy Expert System associating Income and Sensitivities to Price and Quality | 6.29 |
| 6.7 Operations for modifying the Fuzzy Sets depicting the Consumers' Perceptions of Price and Quality | 6.31 |
| 6.8 Rules used by the Fuzzy Expert System associating the price perceived p_{iq} and its correspondent attractiveness | 6.39 |
| 6.9 Rules used by the Fuzzy Expert System associating the quality perceived q_{ig} and its correspondent attractiveness | 6.40 |
| 6.10 Possible Values of the Synthetic Evaluation σ_{igj} | 6.46 |
| 6.11 Maximum and Minimum Values for the Synthetic Evaluation | 6.48 |
| 6.12 Regression Equations for the Extreme Values that can be assumed by σ_{igj} as a Function of s_{pj} | 6.49 |
| 6.13 Consumers' Payoffs as a Function of σ_{igj} and s_{pj} | 6.50 |
| 6.14 Summary of the Game's Payoffs | 6.51 |
| | |
| 7.1 Costs and Prices for Phase I | 7.5 |
| 7.2 Phase II, Group A: Constant Prices and Varying Costs | 7.6 |
| 7.3 Costs, Prices and Markups for Games of Phase II, Group B | 7.6 |
| 7.4(a) Costs, Prices and Markups for Games of Phase III | 7.7 |
| 7.4(b) Costs, Prices and Markups for Games of Phase III | 7.8 |
| 7.5 Results of Firm 0 | 7.22 |
| 7.6 Results of Firm 1 | 7.23 |
| 7.7 Results of Firm 2 | 7.25 |
| 7.8 Results of Firm 3 | 7.26 |
| 7.9 Results of Firm 4 | 7.27 |
| 7.10 Results of Firm 5 | 7.27 |
| 7.11 Total Payoffs received by All Firms and the Population (A) | 7.28 |
| 7.12 Total Payoffs received by All Firms and the Population (B) | 7.35 |
| 7.13 Phase IV: Prices and Costs for Games 1 and 2 | 7.40 |
| 7.14 Determination of the Differentiated Advertising Budget | 7.41 |
| 7.15 (a-b) Game 1 - Equivalent Advertising Budgets | 7.42 |

| | |
|--|------|
| 7.15 (c-e) Game 1 - Equivalent Advertising Budgets | 7.43 |
| 7.16 (a) Game 2 - Differentiated Advertising Budgets | 7.43 |
| 7.16 (b-e) Game 2 - Differentiated Advertising Budgets | 7.44 |

Chapter 1

Introduction

Discovering the Prisoner's Dilemma is something like discovering air. It has always been with us, and people have always noticed it—more or less.

—William Poundstone, in *Prisoner's Dilemma* (1992)

1.1 - Preamble

Understanding the pattern of behavior of *rational agents* when they are confronted by a mutual *conflict of interest* has been deserving a continuous and increasing attention by researchers from various scientific areas. In classical Decision Theory the actors try to find the best solutions to their problems when dealing with *possible states of the world*, that can be deterministic, probabilistic, fuzzy, or even chaotic. That approach usually does not consider specific or particularized reactions from one agent to the deeds of another. On the other hand there is Game Theory, which has been developed aiming at explaining how rational people *ought* to make decisions in antagonic situations, if they want to achieve a particular goal. Game Theory is *normative* and not *descriptive*, in the sense that it does not attempt to make predictions about how the agents will actually behave.

The observed fact that decision makers often fail to follow the theoretical prescriptions regarding their resolutions towards the maximization of the expected utilities raises some concerns about the real meaning of *rationality* and the adequate definition of *utility*, mainly when the dispute cannot be modeled by a *strictly competitive*¹ game. In a non-zero-sum game, the players (decision makers) may not have a strictly opposed

¹ Also called *zero-sum* or *constant-sum* games. See Chapter 2 of this work.

preference of one outcome over another (as in zero-sum games). If both players prefer the same outcome over any other, the former is called *Pareto-optimal* [RAPO92].

In this work, the paradigm of a 2×2 game called the *Iterated Prisoner's Dilemma* (IPD) is explored. Taking into account the vast diversity of individual comportment, that are guided by many factors (psychological, environmental, utilitarian, etc.), a mathematical analytical model of a complex system composed of many PD players turns out to be impracticable. Therefore, this dissertation approaches the problem using simulation techniques, allied to a new fuzzy version of the PD.

1.2 - The Prisoner's Dilemma

The *Prisoner's Dilemma* game (PD), invented in 1950 by Melvin and Dresher from Rand Corporation, is considered by many the quintessence of a conflict of interest, a *social trap* that exposes in an elementary and clear form the discrepancy between *individual* and *collective* rationality.

In the PD, all three principles of individual rational decision, that is, *dominating*, *maxmin* and *equilibrium*, point to the choice "Defection". The players' maxmin strategies intersect in the same outcome, but that is not Pareto-optimal. Nevertheless, the collective rational decision must be Pareto-optimal.

Herodotus² describes an early example of reasoning in the Prisoner's Dilemma in the conspiracy of Darius against the Persian emperor. A group of nobles met and decided to overthrow the emperor, it was proposed to adjourn till another meeting. Darius then spoke up and said that if they adjourned, he knew that one of them would go straight to the emperor and fink on them because if nobody else did, he would himself. Darius also suggested a solution - that they immediately go to the palace and kill the emperor³.

² Cited in [RASM89], p. 38, Note 1.2

³ Darius meant that *defection* (delation) is the move that yields the best payoff—the emperor's gratitude and expected privileges— when adopted isolated in the presence of collective *cooperation* (silence about the plot). But if everyone goes to the emperor to delate the conspiracy—mutual defection, all will be obviously worse off.

1.3 - Description of the Problem and Objective of the Dissertation

The Prisoner's Dilemma is not about prisoners. Its use as an adequate paradigm to model conflicts of interest is mirrored by a crescent number of applications in operations research, economics, biology, sociology, political science, etc. Thus, the achievement of a better insight of the dynamics of the PD game are of great value in predicting how competitive systems might evolve.

The situation described in Herodotus fable above well illustrates the basic nature of the PD. However, those circumstances reflect the dichotomic character of the classical game, in which there are no intermediate actions: A player can only either *cooperate* (C) or *defect* (D). But in real-life, practical problems, that is seldom the case, and the traditional binary PD is not able to capture the nuances of the real processes being modeled. A decision maker can almost ever employ gradual strategies, chosen from a continuous palette of options between two limiting points. And even if the question alone does not admit gradation concerning the final decision, the reasoning process that leads to the concluding move certainly passes through a non-discrete process. Likewise, the world is generally sufficiently diverse to allow a player to compensate for an extreme settlement with another relaxed one.

A question that comes to mind is, what would happen in the IPD if one departs from the traditional dichotomic actions C and D and the participants in the game may select actions in a continuous range? Moreover, what if the players make their decisions based on a qualitative method?

The attempt to offer a contribution to the comprehension of those questions constitutes the main subject of this dissertation, which is the development of a model of the IPD where the foregoing general assumptions are considered. In order to achieve that goal, a version of the PD has been developed and implemented. It has been denominated the *Fuzzy Iterated Prisoner's Dilemma* (FIPD), and relies basically on fuzzy set theory and fuzzy expert systems.

Why use fuzzy sets associated with the Prisoner's Dilemma paradigm?

Before offering a straightforward answer, let us consider the way a person drives a car. Depending on the vehicle trajectory, the driver assesses the situation and reacts steering to the right or left, with different emphasis. In this manner, though the driver *succeeds in maintaining the car under control*, it does not know either the exact angle and other quantitative characteristics of its action conveyed to the wheels by the mechanical system of the car, or the external physical numerical attributes of its path.

In the example above, the driver is using a qualitative method to make its decisions. In the PD, on the other hand, the counterpart of a rational player is another rational player, and they have conflicting (albeit non-strictly) interests. That situation is, in a sense, much more complex, because some variables of very difficult and questionable quantification are present in the decision process (e.g. rationality, preferences, utility, not mentioning moral related attributes). Thus, the process is *blurry* because the variables that take part in it are also blurry.

Nevertheless, to handle this kind of situation, fuzzy set theory (FST) can provide a adequate method to mirror the rules by which people manage and negotiate the perceptions and actions involved in the game. Additionally, it has already been demonstrated by numerous successful applications in many areas that the FST can augment, and eventually exceed other more conventional mathematical techniques regarding the performance of an obtained solution or control procedure directed at nonlinear or other complex phenomena.

Such is the problem of the IPD, and it is assumed, with confidence, that the IPD is the case for an application of fuzzy set theory and fuzzy expert systems, which resulted in the FIPD models.

Two approaches regarding the FIPD are investigated: The first regards a computational tournament of the FIPD where 512 different fuzzy strategists confront themselves and three other peculiar players, namely, *Tit-for-Tat* (TFT), *Pavlov*, and also a fuzzy version of TFT. The second consists of a practical application of the *one-sided* version of the FIPD.

Apart from other specific questions addressed, especially in the latter and more elaborate model, the following topics motivated the research:

- How do the players perceive a opponent's strategy and what are their responses?
- What is the dynamic behavior of the whole environment, concerning the conflict of interest embedded in the process ?
- Can an equilibrium be established? If it can, which are the characteristics of the strategies that compound it?
- Which are the dominant strategies in terms of attaining the best results? Are they stable?

A practical application of the IPD is developed along Chapter 6, where the problem of modeling a competitive market is studied under the one-sided IPD approach. There, an environment with a variable number of Firms (sellers) and Consumers (buyers) has been designed and initialized. Each Firm arbitrates the advertising budget, the price and the cost (associated to the quality) of its product or service, and offers it for sale in the market. The population is composed of one thousand buyers, differentiated by their incomes and preferences relative to the price and quality of an item.

While the Firms can fix their variables freely, which is done arbitrarily in order to investigate the most successful policies, the consumers operate in a highly dynamic mode, where the population's consolidated decisions constantly influence each individual. It also is affected by the tradeoff between its own expectations and concrete results achieved.

The final objective of the model contained in Chapter 6 is finding out how prices, costs and advertising budgets are related concerning the achievement of the best economic results, not only in terms of the market share but also of the net profit. It is important to note that the consumers are not *static* in their decisions, because they are also rational in the sense of seeking the greater benefits when dealing with the sellers. The subsequent

chapter implements the *One-sided Fuzzy PD*(1SFIPD) through a simulation program with several examples.

1.4 - Methodology

The performance of strategies in the dichotomic IPD has already been exhaustively studied. The computational tournaments of the IPD accomplished by Axelrod [AXEL84] greatly increased the knowledge about how cooperation evolves, and which are the characteristics of the most effective strategies to be employed. But all those conclusions are entirely **context dependent**, and further research demonstrated that some very successful strategies, like TFT, would not do so well if some other newly formulated rules (like Pavlov, or Generous TFT)⁴ were included among the opponents. Even so, the alluded tournaments relied entirely on previously formulated plans by human strategists.

The current direction of the research in the area is to employ *soft computing* and artificial intelligence techniques allied to intensive simulation methods in order to automate the process of discovering new tactics to successfully playing the game.

In Chapter 5, the FIPD is outlined. Each player is represented by a collection of three fuzzy expert systems (FES). The FES have as inputs, or antecedents, three distinct factors, which regard the relation between a player and its opponent's *wealth* (f_1), the adversary's last three moves (f_2) and the relation between the *trends* of its accumulated payoff and that concerning the entire population. The outputs are given in terms of the gradual action to be implemented in the current move, which varies between *zero* (total defection) and *one* (total cooperation). The players using that decision process add up to 512, as a result of the combination of the different tactics employed for each factor. Included in the participants are three other well known strategists, previously mentioned. A computational tournament using a program in C++ language especially developed for the simulations was accomplished, and several conclusions were drawn from the results [BORG95].

⁴ See Chapter 3.

The primary purpose of that part of the dissertation is to investigate the performance of the fuzzy players regarding their particular strategies. Because of the great number of possible pairwise combinations (132355), the contestants were randomly divided in groups, each confronting a specific opponent about 150 times, on the average.

The essence of the model included in Chapter 5 is fundamentally investigative, and was developed as an inquiry about the effects of qualitative reasoning in the IPD.

The method utilized in Chapter 6 this work is rather distinct. Departing from the usual recent approach of automating discovery of new effective strategies by means of AI techniques [AXEL87], [FOGE93a], [FOGE94c], the procedure had a converse goal: How can a player effectively choose its strategy when randomly confronted to adaptive, AI guided players? Observe that the latter do not confront each other, so the iterations take place only between the two types of players present in the planned market share game, namely Firms and consumers. The paradigm employed was the one-sided IPD [RASM89], in which only one type of player—the Firms—is involved in a PD-like situation. The iterations consist of randomly assigned encounters between buyers and sellers⁵. Of course, the seller would prefer to sell a low cost item for a high price, and many of them. On its turn, the buyer generically favors the value of its acquisition, choosing to buy the best quality product or service for the least amount of money. A conflict of interest is therefore formed.

The Firms may select their actions (price, cost and advertising budget) in a continuous interval, which was manually done, aiming at finding the relations between the selected policies and their performance. On the other hand, the Consumers, like in the classical PD, have only two choices: buy or not from a given supplier. But in the model, the Consumers are not at all equal in their resources and tastes, and they perform constant updates in their decision processes, which are again based in fuzzy expert systems and Belief theory.

⁵ The mechanisms used to pick a pair of players to interact are not random in the sense of “any pair is equally alike”, but instead, depend on the *advertising budget* specified by each Firm.

The game is divided into *cycles* where a fixed number of iterations is completed. During each cycle the Firms' variables remain constant, but they may be adjusted for the next round.

A simulation C++ program has been developed to implement the iterations, and the results are presented, discussed and analyzed.

1.5 - Structure and outline of the Dissertation

The dissertation has been divided in eight chapters. Chapters 2 through 4 contain a bibliographic review of the topics of the theory related to the development of the research. Chapters 5 through 7 approach the Fuzzy Iterated Prisoner's Dilemma, new with this work and constituting the core of the project. Chapter 8 presents the conclusions.

In the sequence, a brief description of each chapter's content is given.

- **Chapter 1:** This introduction, where the outline of the work is disclosed, including the motivation, essence of the problem approached, and the methodology employed.
- **Chapter 2:** Contains a review of the fundamental aspects of Game Theory. The text includes the characterization of a conflict of interest as a strategic game, competitive (strictly and non-strictly) and cooperative games. Also discussed are the concepts of rationality, dominance, equilibrium and arbitration schema.
- **Chapter 3:** Embodies the analysis of the Prisoner's Dilemma game, which deserved a particular attention considering the fact that it constitutes the cornerstone of the research. The history, applications and most important theoretical developments of the PD paradigm are examined and detailed. Special mention is made to the computational tournaments of the iterated PD, as well as quite recent works using artificial intelligence (AI) techniques for the modeling of the problem. To complement the review, the chapter includes topics of other significant PD-like games, as the biological-oriented *Hawk-Dove*.

- **Chapter 4:** The model developed in the Dissertation compounds the PD with Fuzzy Set Theory and respective tools. Therefore, a review of the related techniques has been judged necessary. The chapter provides a description of the methods relying on fuzzy logic, fuzzy measures and belief theory, which have been taken advantage of in the formulation of the FIPD. Moreover, the working of the *fuzzy integral* is presented, and a modification in that method is introduced.
- **Chapter 5:** An investigative version of the Fuzzy Iterated Prisoner's Dilemma is detailed and simulated with a computer program. The strategies employed by the players depart from the traditional binary restriction, and the process of picking a move relies on fuzzy expert systems that take in consideration inputs other than the usually adopted *sequence of the opponent's last moves*.
- **Chapter 6:** Consists of a practical application of the *one-sided* version of the FIPD. In order to supply a realistic environment for the analysis of a conflict of interest modeled by the FIPD, a market share game including buyers and sellers has been designed, where the contestants exercise their strategies. The decision process of the buyers has been intended to mirror, as far as possible, the behavior of rational agents adopting qualitative reasoning. The experiments are accomplished by means of a object-oriented C++ program, implemented by intensive simulation of the interactions that take place between pairs of the agents, the Firms and the Consumers.
- **Chapter 7:** Includes a description of the program utilized to perform the experiments, as well as the results attained by the computational simulation of the market share game detailed in the previous chapter. The outputs derived from the various situations assessed in the simulations are extensively examined, along with a graphical presentation of the most outstanding results.
- **Chapter 8:** Conclusions. An outlook of the research is depicted, including limitations of the work and suggestions of topics for further development.

1.6 - Main Contributions of the Dissertation

The most important advancements contained in this work regard the following topics:

- Use of fuzzy reasoning in the Prisoner's Dilemma Game;
- Introduction of an adjustment process in the traditional fuzzy integral;
- Development of a method of aggregating multiple decision criteria using the modified fuzzy integral and elements of belief theory;
- Implementation of an application of the 1SFIPD to a practical problem;
- Demonstration of the feasibility of using the IPD as a basis of building tools for decision-making through simulation techniques.

Chapter 2

A Review of Fundamental Topics of Game Theory

2.1 - Introduction

The purpose of this chapter is to provide a background of the most vital and basic aspects of Game Theory. It is usually accepted that the mathematically formal study in this area began with Von Neumann's papers published in 1928 and 1937 [NEUM28], [NEUM37]. Another author, Maurice Fréchet [FREC53], considers that this initiative should be credited to Emile Borel [BORE38], although with some exemptions¹. The book "Theory of Games and Economic Behavior" is habitually referred as the first complete and systematic approach to the subject [NEUM44]. Game theory deals with situations where two or more agents have some conflict of interest, about some limited or scarce resource. It is not a prescriptive theory, because it does not intend to tell how people behave or make decisions, neither make predictions about their acts. If it were so, such a theory would be also descriptive. The accuracy of a descriptive theory could be measured by the rate of success with which it foresees what will be done in some particular circumstances by rational agents. So, rather than descriptive or prescriptive, game theory aims to be normative.

The goal of a *normative* theory is to inform rational people what they ought to do to achieve the desired ends. Here a problem arises: What is it to be rational or not? The concept of rationality is commonly associated to maximization of gains, which, by its turn seems to imply an egoistic and selfish conduct. Anatol Rapoport, from University of

¹ The exemptions regard the non-achievement of a proof for the Minimax Theorem, which is viewed as a cornerstone of Game Theory. Borel supposed that the Minimax Theorem was not generically valid, being only applicable in special circumstances. The merit of proving it for general conditions is owed to Von Neumann.

Toronto, presented a very good discussion of this theme [RAPO90], [RAPO92]. It is argued that there is nothing wrong with a gains-maximizing strategy. There are a whole lot of things, concrete and abstract, that can be categorized as *gains*.. It can be money, power, self-esteem, knowledge, recognition, etc. On the other hand, one less disputed definition, is that of what consists a *rational agent*: It is someone who acts bearing in mind what the consequences of its decision should be. However, to accomplish this objective, the agent must be able to effectively choose among the alternatives, be aware of which consequences each one of them will entail, and, ultimately, have the aptitude to establish its preferences over the available choices.

Before the development of Game Theory, the theory of decisions under risk was already established. Differently from the former, which is characterized, among other important things, by the existence of multiple (two or more) intelligent agents, Decision Theory involved only one actor and an uncertain situation, or probabilistic environment. This situation is sometimes mirrored as “games against Nature”.

In that setting, the notion of risk arises from the fact that the decision maker cannot, alone, or exclusively by means of its own acts, determine an. The result will be a consequence of its acts and of the “state of the world” that comes forth, and can be classified, according to the agent’s preferences, in an ordinal scale, from better to worse.

If decisions are to be made under certainty, all that has to be done by the agent is a specification of the qualitative order of its preferences, with no quantitative measures attached to them. But, given the stochastic feature of the environment, with probabilities associated to the occurrence of events, a numerical scale must be introduced regarding the possible outcomes. The combination of these pairs of quantities is defined as the *expected gain*, which, in short, is a weighted sum of all possible gains that correspond to a particular decision, where the weights are the probabilities of each occurrence. This concept has been established in the mid-seventeenth century, by Blaise Pascal and Pierre de Fermat². Here, the same normative feature present in Game Theory is also extant, because the method of

² The rigorous definition of mathematical expectation, that underlies the idea of Expected Gain, was first given by Christian Huygens, in “The Rationalis in Ludo Alea” . This work, published in 1657, contains the correspondence exchanged between Pascal and Fermat about this matter (Op. cit. [RAPO90]).

the expected gain did not intend to predict behaviors of the agents, as a descriptive theory should. Empirical evidence showed that human agents hardly ever made decisions that conformed to what should be called *rational*, that is, gains maximizing. The discrepancies observed between the prescriptions of rational and actual choices were also extended to what should be understood as “commonsense behavior”.

A very well known and dramatic instance of the foregoing situation is reflected by the St. Petersburg Paradox³. The game contained in the St. Petersburg Paradox shows the inadequacy of the method of maximum expected value as an explanation for rational behavior. Hence, instead of value, what should be maximized is the *utility*, which is defined as a function that assigns a cardinal measure to an individual’s preferences.

Although the theory regarding individual games can be defined as the analysis of situations where a conflict of interest between players is present, the conflict in itself does not have a significant role in the modeling of the players’ comportment, but actually in the individual results that are consequences of those comportment. In that case, the classical theory presumes that:

- i. The possible outcomes are perfectly characterized and known by all players;
- ii. The available information is common knowledge among the players;

Additionally, it is assumed that every individual has ordered preferences for the feasible outcomes, and that each also recognizes the other players’ rankings. The preferences are represented numerically by an *utility function* for each player, which has as arguments the variables that compound the possible results of the game.

The *utility* concept will be discussed in a subsequent section of this chapter, which also examines other relevant themes of Game Theory, with the following structure:

- Representation of Strategic Games;
- Information Sets;

³ The St. Petersburg Paradox is a game proposed by Daniel Bernoulli in the mid-eighteenth century, where the maximization of the expected value criteria seems to lead to an irrational decision. He also proposed an explanation for the paradox, suggesting that the utility of money is not a linear function of the amount obtained. For further information about the paradox, see [SAVA72].

- Strategies;
- The Minimax Theorem;
- Preferences, Utility and Rationality;
- Extensive and Normal Forms;
- Non-zero Sum Games;
- Nash and other Equilibria;
- Cooperative Games and Arbitration Schemes.

2.2 - Representation of Strategic Games

Strategic games are distinct from *parlor games* because their outcomes are not entirely dependent on random occurrences. In the former, the players' decisions influence or may even determine gains or losses to be achieved at the end of the game.

A strategic game can be represented in two ways:

- a. As a connected graph, or *decision tree*, with **nodes** that symbolize decision points along the game, and **branches**, which indicate the possible decisions that can be adopted from each node.
- b. By a complete collection of rules, that establish which player moves next, and the feasible alternatives, for every situation that might take place.

2.2.1 - Decision Trees

The decision trees that represent games have a finite number of nodes and branches⁴. The nodes of a graph may admit *antecedents* and *successors* and are organized in various levels. A node without successors is called *terminal*, and the starting node is named *root*. A *move* is the decision made by a player, depicted by a branch. A *match* is the players' sequence of moves. A particular stage of the match is denoted by a *level*. Level 0 is the root node.

The moves that have been reached by different paths along the graph are discriminated. When a player finds itself in a node, it should implement its move. For that

⁴ A distinction should be made between this condition and the number of repetitions of a game, that can be infinite.

purpose it relies on an *information set*, which stands for the maximum available knowledge to the participant at that stage according to the rules of the game. The extension of the information set is a valuable advantage, for it includes the data that is legally allowed by the rules that, which will aid the player' choice⁵.

The players might be interested in exploring the game's graph, so that they may get aware of the most advantageous moves. The most common investigation techniques are the *width* and *depth* explorations. The first hypothesis consists of an identification of all nodes of a given level i , before examining level $i+1$. In the depth exploration, the survey proceeds from a node j belonging to level i to another node k pertaining to level $i+1$, until a terminal node is reached. When this task is finished, the search goes back to the previous level, and the process is repeated until all terminal nodes are found.

It is convenient for a player to have a method to quickly compare possible sequences of moves, indicating those most promising in terms of future acquisition of favorable positions. The expression favorable positions is related to a multicriteria decision problem, since it may regard potential gains, relative advantage, reduced exposition to losses, more diversified oncoming choices, etc. The aggregation of all those principles can be likened to a player's utility function, but its assessment would require an extensive search of the decision tree, and this is just what is opportune to avoid.

One viable way to establish the priorities for exploring the game's graph is to employ a *sub-optimal function* [SAMP76], which allows the player to predict, with some statistical confidence, the consequences of an action, spanned to the future levels of the game.

2.2.3 - Rule Sets

Depending on the number of nodes, levels and branches, it may be the case that the representation of a game by means of a graph becomes impracticable. The decision tree for Chess, for instance, is estimated to have 10^{120} nodes. So, a game can also be described by a set of rules, which must completely and unambiguously specify the development of the

⁵ One way to cheat in the game is to take possession of illegitimate information.

game departing from every node. The antecedents, successors and the termination criteria⁶ must also be defined. The methods of exploration mentioned for the graphical representation are also applicable in the current case.

2.3 - Usual Methods for the Evaluation of the Sub-optimal Function

2.3.1 - The Minimax Method (or Principle)

This is one of the most often used methods to measure the advantage of a set of decisions in a game. Note that an evaluation method is as good as far it can “look ahead”, which corresponds to the number of future levels that it considers.

In a game with two players, the Minimax method assumes that one of the participants (say, the first), will attempt to achieve the greatest benefit for itself, whereas the other will try to confine that benefit to the smallest possible amount.

Consider, for example⁷, the graph depicted in Figure 2.1. Levels {0; 2} and {1; 3} contain the nodes where the first and second can be, respectively, in any instant of the match. The branches indicate their possible moves. In the decision tree, the nodes of Level 3 show the payoffs, that were estimated by an arbitrary evaluation function.

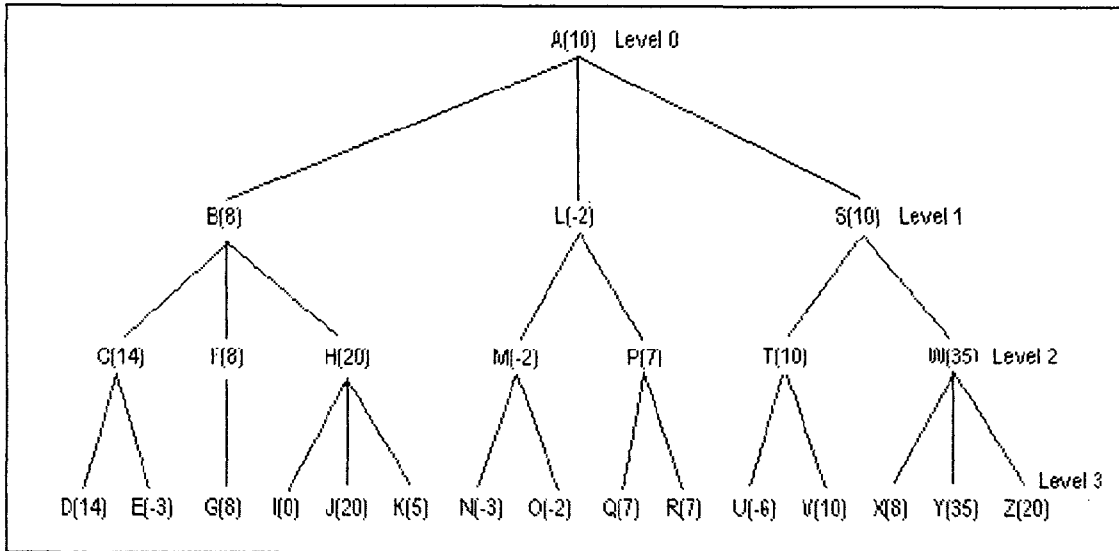


Figure 2.1 - Fragment of a Decision Tree with the Nodes' Gains attributed by the Minimax Method

⁶ A particular termination condition may be the number of levels.

⁷ The example was adapted from op. cit. [SAM76].

The current level is *zero*, where player 1's decision shall be made. The gains associated to the nodes of Levels 2, 1 and 0 were obtained using the Minimax procedure. The method attaches to the first player's nodes the highest possible payoffs, and to the second player's nodes, the lowest gains. In Level 1, for instance, the first player's gain can be limited by the second to 8, or -2 or 10, in case the first plays B, L or S, respectively. But if it takes into account that the other employed that reasoning, then the first should opt for S, that will assure it a gain of at least 10. A variant of the same method is the *Minimum Regret Principle* [BLAC54].

2.3.2 - Alpha-Beta Cuts

The Minimax procedure can be improved by reducing the extension of the exploratory search in future levels. One common way of accomplishing this objective is using a process called *alpha-beta cuts*.

Assuming that the first player opted for move S, from Figure 2.1, it can be deduced that the second player will favor T, discarding W, because this latter option implies in a greater gain for the first player (35) than the choice T, which yields only 10. When the available situations are investigated from W, one finds the value 35 in Y, higher than in any other node, and of course the first player, if allowed to decide from W, would not rationally select either X or Z. With this reasoning, the need to examine X, Z, and their successors is eliminated. The value 10 in node W is denominated *beta*, which corresponds to the *minimum of the maxima* found among the successors of a node.

The value of *alpha* is similarly determined, but then examining the decision levels concerning the second player and picking the *maximum of the minima*. The reason behind this procedure is that the second player is assured that the first would not select an option that might result in a lower outcome. Referring to the graph of Figure 2.1, the exploration of branches L and B from S turns out needless.

The combination of those two mentioned processes is designated an *alpha-beta cut*. It can be further enhanced by ordering the gains associated to the nodes, so that those with better chance are investigated in first place.

2.3.3 - Other methods

Using heuristics, it is possible to reduce still more the number of nodes to be weighed, by the introduction of an estimate of the probability that the optimal move is discarded from the analysis when some nodes are not considered. With the same goal, that is the decrease of the size of the region in the graph to be contemplated, the methods called *tapered forward pruning* and *convergence forward pruning* can be cited⁸.

2.4 - Information sets

An information set is the amount of knowledge allowed to a player in a generic moment of a game, according to the game’s rules. It is depicted by a set of different nodes in the decision tree. Basically, in a game of strategy, without any random factors, a player is able to know only the level where it is, but not the specific node. The smallest the collection of nodes of an information set, the greater is the amount of information available to a player. An important restriction regarding the information sets is that for the nodes in the same set, the number of alternatives departing from it must be the same, for if it were not so, the player could discriminate the particular region where it currently is by simply counting the number of existent alternatives, therefore reckoning the sub-region occupied.

Regarding the information structure, a game can be characterized in four different ways [RASM89]:

| Information Category | Types |
|----------------------|-----------------------------------|
| 1. Perfectness | 1.a) Perfect 1.b) Imperfect |
| 2. Certainty | 2.a) Certain 2.b) Uncertain |
| 3. Symmetry | 3.a) Symmetric 3.b) Asymmetric |
| 4. Completeness | 4.a) Complete 4.b) Incomplete |

Table 2.1 - Information Categories of a Game

⁸ A more elaborate discussion about those methods is beyond the scope of the present Chapter. See [JACK74].

- ♦ **Perfect information games** are those where each information set has only one node, that is, it is a *singleton*. In these games a player knows its exact position in the decision tree. The moves are sequential, not simultaneous, and any eventual random moves played by Nature are correctly observed by the players.
- ♦ In **certain information games**, Nature does not interfere (make any moves⁹) after any player has moved. On the other hand, in a game of **uncertainty**¹⁰, Nature's moves may or may not be revealed to the players immediately.
- ♦ In a game of **symmetric information**, every player's information set has the same elements at any decision node or at the end node⁹. Under this category, Nature can make its moves and concurrent progress is also allowed, although no player can have privileged information. In the **asymmetric type**, a player might detain some **private information**, in the sense that its partition data is different and not worse than another player's.
- ♦ **Complete information games** are those where Nature does not move first, or, if she does, the event is observed by the players. If a game is of **incomplete information**, it must be also of imperfect information, because in the former case some player's information set is not a singleton anymore.

An important concept regarding information sets is that of *common knowledge*. When this property is present in a game, any participant, besides being acquainted with the decision tree, also knows that the others have the same learning¹¹.

2.5 - Strategies

Along a game's match, the contenders make decisions that determine an unique path in the graph, which goes from the initial to an end node. The collection of decisions that each player i makes is called a **strategy** s_i . In other words, a strategy s_i is a contingency

⁹ A Nature's move configures what is denominated a *state of the world*.

¹⁰ The definitions "game of uncertainty" and "games of asymmetric information" have been introduced by Rasmusen in op. cit. [RASM89].

¹¹ The term common knowledge is employed to avoid an infinite recursion of the type "each player knows that the others know that the others know...".

plan that describes which action to take in every circumstance, from the beginning to the end of the game.

A game can be played of several manners, depending on the strategy s_i selected by every participant. Each player possesses a strategy set S_i that includes all possible strategies that can be implemented. Then, a match will consist of a selection of adopted strategies, or a strategy combination $s=\{s_1, s_2, \dots, s_m\}$ that implies in the outcome of the game. This definition refers to strategic games, those where there is no *chance* involved.

2.5.1 - Saddle Points

Consider a strategy game with two players, P_1 and P_2 , which have strategy sets with m and n elements, respectively. The outcome of such game is described by one of the $(m \times n)$ cells from the matrix $A = (a_{ij})$, where each cell corresponds to the payoffs to be obtained by P_1 and P_2 when they employ the strategies i ($i = 1, 2, \dots, m$) and j ($j = 1, 2, \dots, n$), in that order. Suppose that the payoffs represent payments that P_2 must make to P_1 . Negative values mean that P_1 should pay an amount to P_2 . Each participant is seeking the highest feasible gain, what means the maximization of a_{ij} for P_1 and the minimization of the same quantity for P_2 . Given a strategy i chosen by P_1 , that player knows that its payoff is at least $\min_j a_{ij}, j=1, 2, \dots, n$. The criterion for the singling out i is the obtainance of $\max_i \min_j a_{ij}, i \leq m, j \leq n$. Analogously, P_2 can anticipate at least $\max_i a_{ij}, i \leq m$, and therefore impose a limit of $a_{ij}, i \leq m$ to player P_1 , for any strategy j . Because P_2 's goal is symmetric in relation to P_1 , it will choose j so that it can obtain $\min_j \max_i a_{ij}, i \leq m, j \leq n$.

The gains expressed by $\max_i \min_j a_{ij}$ and $\min_j \max_i a_{ij}$ are connected to the strategies picked by P_1 and P_2 . It can be proved that those quantities obey the following relation[DRES81]:

$$\max_i \min_j a_{ij} \leq \min_j \max_i a_{ij} \quad \text{Eq. 2.1}$$

It may also happen that the inequality in 2.1 is transformed in an equation, that is:

$$\max_i \min_j a_{ij} = \min_j \max_i a_{ij} = d \quad \text{Eq. 2.2}$$

In those circumstances, P_1 can always select an strategy i^* such that guarantees for itself a gain of at least d . Conversely, P_2 is capable of picking another strategy j^* that averts P_1 from receiving more than d . If Equation 2,2 verifies, i^* and j^* are optimal strategies for P_1 and P_2 , which correspond in matrix A to the cell $a_{i^*j^*}$. That element is denominated the *saddle point* of the game and assumes the value d .

The optimal strategies i^* and j^* have the following properties:

- P_1 will always be able to obtain at least the payoff d if it chooses i^* , independently of P_2 's choice.
- Playing j^* , P_2 is in a position to confine P_1 's gain to at most d , no matter which strategy P_1 implements.
- If either player previously announces that it will play its optimal strategy i^* or j^* , this fact does not bring any relevant information to the other, in the sense of supplying it with additional advantages (raising or decreasing d , respectively for player P_1 and P_2).

The existence of saddle points is a characteristic of games with perfect information, which may or may not be strictly strategic¹².

In the hypothesis that *chance* moves occur, each player, when deciding about its move, should take into account the possible outcomes from the random process, as well as the consequences of the chance moves regarding the game's development¹³.

2.4.2 - Pure and Mixed Strategies

An strategy is called *pure* when the players' actions are deterministic and perfectly specified for every possible conjuncture to appear during the game. Before its move is implemented, any player occupies a position in a particular level of the decision tree, within a certain information set. To play, the participant has to opt for one of the branches that depart from the node where it currently is. Given that each generic player i may be in one of k information sets and discriminating by q ($q = 1, 2, \dots, r$) the branch it chooses, a pure

¹² Another form of characterizing games with perfect information refer to the knowledge that the players detain, in each of their moves, of the game's history: The available data must be complete, retrievable and with no ambiguities.

¹³ Depending on the complexity of the game, regarding the quantity of nodes, levels and branches, the optimal strategies, though theoretically existent, many times cannot be found, because of the combinatorial expansion of the alternatives when an in-depth search is performed. Chess is a common example of that situation.

strategy s_i is a another set with k elements, each defining the action q^i to be accomplished by the player for every one of the k information sets.

Generically, for n players, the collection of pure strategies that can be employed in a match is $S_n = \{s_1, s_2, \dots, s_n\}$.

Nevertheless, when it adopts a pure strategy, a player is granting additional information to the others, since each will be able to spot, by the analysis of the foregoing moves, in which node of the current level that pure strategist is located. That implies in a deliberately conceded advantage to the opponents, which is a circumstance to be avoided in competitive games. To face that problem a player can execute a procedure designated by **mixed strategy**. This method consists in selecting moves dictated by a random process. In this manner a player will elude its adversaries by barring them from recognizing its strategy pattern¹⁴. Note that not even the own player will be able to know by anticipation which its next move will be.

Therefore, the collection of k actions with respect to the formulation of a pure strategy needs to be modified, because it is not feasible to have a previous comprehensive description of every player's reactions to the different situations that may take place in the game.

A mixed strategy for a generic player can be expressed by a matrix $X = [x_i]$, with one column and as many rows as the number of alternatives for the moves. The elements x_i of the matrix X are the probabilities that the i -th strategy is employed, $i=1, 2, \dots, m$. Thus, X stands for the probability distribution of the m strategies of a player. Each player will have its own particular matrix, inasmuch as both the strategies and the probability distributions are specified by the player itself.

¹⁴ For the adequate implementation of the method, it is necessary that the moves be decided by means of a randomizer. It is a well known fact that mental alternation of decisions is pseudo-random, always containing some kind of regularity that may be detected by various mathematical techniques.

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = [x_i], i=1, 2, \dots, m; 0 \leq x_i \leq 1; \sum_{i=1}^m x_i = 1$$

The mixed strategy represented by X turns into a pure strategy when any of its elements is the unity, hence implying that all other are zero. So, only one strategy subsists.

How should a player elect the probabilities x_i ? A possible answer is to maximize its *security level*. For clarity, let us consider an example.

A game with two players has the following payoff matrix:

$$\begin{array}{cc} & \beta_1 & \beta_2 \\ \alpha_1 & \begin{bmatrix} 2 & -6 \end{bmatrix} \\ \alpha_2 & \begin{bmatrix} -3 & 5 \end{bmatrix} \end{array}$$

This game does not have a saddle point, because no single element of the matrix is the minimum of a row and the maximum of a column at the same time. If the row player (P_α) picks α_2 , its security level is maximum, equal to a loss of -3, as opposed to α_1 , that makes it vulnerable to the greater loss of -6. With an equivalent purpose, P_β should play β_1 . However, when P_α performs this analysis, it is tempted into playing α_1 , which brings it better benefits when combined with β_1 . On its turn, P_β also reasons in the same direction, thereby preferring β_2 . This tends to bring up a never-ending chain of meta-reasonings with no final objective conclusion other than the existence of an *indifference* of P_α between the two possible moves α_1 and α_2 . To choose, P_α may employ a random device with two equally probable outcomes¹⁵. P_β might also do the same.

¹⁵ An interesting question is: If a player is indifferent to either strategy, then why use a randomizer, instead of simply arbitrating it? The reason may lie in the fact that a random device supplies the opponent with no specific clues, that could be built on discovering an opponent's eventual personality bias. Furthermore, a player might also want to allow different probabilities of occurrence to the outcomes, which is obviously impracticable to be mentally implemented.

A *mixed strategy* is therefore defined as a method of selecting moves by associating pre-defined probabilities of occurrence to each of the available *pure strategies*. With this definition as a basis, the Minimax Theorem is established.

2.5 - The Minimax Theorem

As mentioned in the preceding section, in zero-sum games where a saddle point exists, it corresponds to a pair of pure strategies i^* and j^* such that $\max \min a_{ij} = \min \max a_{ij} = d$, for $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$. If that condition verifies, then the strategies i^* and j^* are said to be an equilibrium¹⁶. Conversely, if two strategies i^* and j^* are in equilibrium, it means that there is no other group of actions i^0 that is a better response than i^* to j^* . The same is valid concerning j^* .

Now let us contemplate a game without a saddle point, or equilibrium pair. In this case both P_α and P_β will employ each a particular mixed strategy, that is a probability distribution over the pure strategies α_i and β_j respectively. The column matrices $M=[x_{\alpha i}]$ and $N=[x_{\beta j}]$ stand for the probabilities that the pure strategies α_i and β_j are adopted.

Assuming that P_α has selected a particular strategy i and that $A=[a_{ij}]$ represents the payoff matrix of the game, the expected gain $g_{\alpha i}$ of player P_α is given by Equation 2.3.

$$g_{\alpha i} = \sum_{j=1}^n a_{ij} \cdot x_{\beta j} \quad \text{Eq. 2.3}$$

Thus, for P_α , the expected gain depends on which strategy i it decides to employ, being depicted by the i -th row of the column matrix G_α .

$$G_\alpha = [g_{\alpha i}] = [a_{ij}] \times [x_{\beta j}] = A \times N \quad \text{Eq. 2.4}$$

Analogously for P_β , its expected gain when practicing a strategy j against P_α 's mixed strategy is

$$g_{\beta j} = \sum_{i=1}^m a_{ij} \cdot x_{\alpha i} , \quad \text{Eq. 2.5}$$

¹⁶ See section 2.9.

which corresponds to the j -th element of the row matrix G_β .

$$G_\beta = [x_{\alpha i}]^T \times [a_{ij}] = M^T \times A \quad \text{Eq. 2.6}$$

If both players employ mixed strategies, the expected gain of the game, taken as P_α 's payoff, is given by Equation 2.7.

$$E(g_\alpha) = M^T \times A \times N = \sum_{j=1}^n \sum_{i=1}^m a_{ij} \cdot x_{\alpha i} \cdot x_{\beta j} = G_\beta \times N = M^T \times G_\alpha, \quad \text{Eq. 2.7}$$

where M^T is the transposed matrix of M .

While the players can have control over the parameters that they picked for the probability distributions concerning their respective mixed strategies, the effective sequence of moves shall be a consequence of the global random process. If P_α selects a mixed strategy M , its final payoff in the game will be at least $\min_N M^T \times A \times N$. Because P_α is seeking the highest gain, it will choose the mixed strategy M^* , so its payoff will become $\max_{M^*} \min_N M^{*T} \times A \times N$. Symmetrically, P_β 's goal is to make P_α 's gain as small as possible, thus choosing a mixed strategy N^* . The advantage that P_α is able to achieve is bounded by P_β 's tactics and equal to $\min_{N^*} \max_M M^T \times A \times N^*$.

It can be proved that

$$\max_{M^*} \min_N M^{*T} \times A \times N \leq \min_{N^*} \max_M M^T \times A \times N^*. \quad \text{Eq. 2.8}$$

The Minimax theorem¹⁷, which is considered one of the most important results of Game Theory, establishes that there will always exist a value γ that satisfies simultaneously the intents of both players, that is,

$$\max_{M^*} \min_N M^{*T} \times A \times N = \min_{N^*} \max_M M^T \times A \times N^* = \gamma \quad \text{Eq. 2.9}$$

¹⁷ For a proof of the Minimax Theorem, see op. cit. [LUC57] p. 391, or [DRES81].

2.6 - Preferences, Utility and Rationality

Although Utility Theory does not refer exclusively to the Theory of Games, it is known that its modern version was developed by Neumann and Morgensten [NEUM44] as a background to the latter.

The concept of utility and its respective maximization represented a progress towards the characterization of what consist a rational conduct. Even so, the controversy has not dissipated. As the meaning of *rationality* interests such diversified areas as Philosophy, Psychology, Economy, Sociology, the underlying concept of utility has become a topic of ardent discussions among prominent scientists of these areas. Several questions arose from the discussions, but above all, remained these two issues: First, how does a rational agent builds its utility function, thereby being able to assign a numerical value to the set of events gains or losses under consideration? Second, provided that a person possesses, in a way or another, a successfully constructed utility function regarding its predilections, is it always rational to make decisions using the criteria of expected utility maximization?

Before addressing the second point, it is necessary to approach the methods of determining the cardinal utility scale for an agent. This topic was first examined by Neumann and Morgenstern [NEUM44]. Basically, their method relied on the preferences manifested over risky choices, or more specifically, *lotteries*. To illustrate their technique, consider an individual who has ordinal preferences denoted by $R_2 \succ R_1 \succ R_3$ is confronted with three mutually exclusive choices: A or B. A designates the sure-thing non-monetary prize, R_1 ; B is a lottery ticket which enables the player to receive either R_2 or R_3 , both also non-monetary prizes. Whether the lottery yields R_2 or R_3 depends on a probability distribution over these two alternatives. The agent, before determining its pick, will be interested in knowing what are the odds of getting R_2 or R_3 . Assume the probabilities are p and $(1-p)$, respectively. Furthermore, let 0 be the utility that corresponds to R_3 , the least desired outcome, and 1, to R_1 .

Starting from the principle that the agent is acting rationally in the sense that it will choose the option that maximizes its expected utility in situations involving risk, Equation 2.10 follows:

$$u(R_1) = pu(R_2) + (1-p)u(R_3). \quad \text{Eq. 2.10}$$

The term $u(R_1)$ stands for the utility assigned to R_1 , and because R_2 was set to 1 and R_3 to 0, it can be said that p turns out to be the utility associated with R_1 . The utilization of that method presupposes p being varied until Equation 2.10 verifies. At that point, it is said that the agent is indifferent between the exclusive choices A and B.

The success in using the described process is subject to some restrictions. Mainly, there must be a consistency among any pairing of choices. For example, if a fourth alternative is introduced, say, R_4 , the same procedure can be used to determine its utility, substituting R_4 for R_1 in Equation 2.10. If doing this results in $p' = u(R_4)$, the same conclusion must also obtain from Eq. 2.11, where $u(R_4)$ is now derived by means of a lottery including R_1 and R_3 .

$$u(R_4) = (1-q)u(R_3) + qu(R_1) \quad \text{Eq. 2.11}$$

Since it is known from the previous computation and assumption that $u(R_1) = p$ and $u(R_3) = 0$, it follows that $p' = pq$ must be satisfied in order to provide the desired consistency to the method just described. Unfortunately, it is not always the case.

2.6.1 - Fundamental Axioms of Utility Theory

◇ Axiom 1: Ordering of Alternatives

Between any two prize values A_i and A_j it is always possible to establish a preference order, indicated by either $A_i \approx \succ A_j$ or $A_j \approx \succ A_i$. Both preference and indifference are considered as transitive relations so that if $A_i \approx \succ A_j$ and $A_j \approx \succ A_k$, then $A_i \approx \succ A_k$.

◇ Axiom 2: Reducibility of composed lotteries

It is always possible to establish a relation of indifference between any composed lottery and a simple lottery which prizes are A_i , with the respective probability of

occurrence equal to p_i , $i=1, 2, \dots, r$. The relation is expressed by Equation 2.12, where L^j are simple lotteries of the type $p_r^j A_r$, and q_j are the probabilities of their occurrence.

$$\{q_1 L^1, q_2 L^2, \dots, q_s L^s\} \approx \{p_1 A_1, p_2 A_2, \dots, p_r A_r\} \quad \text{Eq. 2.12}$$

◇ Axiom 3: Continuity

In a lottery with prizes A_1, A_2, \dots, A_r ¹⁸, a number $k_i \in [0, 1]$ exists such that for any A_i the indifference relation of Equation 2.13 verifies.

$$A_i \approx [k_i A_i, (1 - k_i) A_r] = \tilde{A}_i \quad \text{Eq. 2.13}$$

It should be observed that the indifference denoted by $A_i \approx \tilde{A}_i$ does not mean that those two quantities are equal. Quite on the contrary, they are two rather distinct values.

◇ Axiom 4: Substitution

Given that $A_i \approx \tilde{A}_i$, \tilde{A}_i can be substituted for A_i in any lottery L , yielding:

$$(p_1 A_1, \dots, p_i A_i, \dots, p_r A_r) \approx (p_1 A_1, \dots, p_i \tilde{A}_i, \dots, p_r A_r). \quad \text{Eq. 2.14}$$

Axiom 3 and 4 taken conjointly form another axiom, denominated *independence of irrelevant alternatives*. It implies that the indifference between A_i and \tilde{A}_i is valid not only when they are secluded, but also when the former is replaced by the latter in any compounded lottery. Stated in another way, any other alternatives $p_j A_j$ present in a lottery altogether with A_i are immaterial in relation to the mentioned substitution¹⁹.

¹⁸ It is assumed that the preferences are ranked $A_1 \approx > A_2 \approx > \dots \approx > A_r$.

¹⁹ Regarding social choices, the independence of irrelevant alternatives is the Condition #3 of Arrow's *Impossibility Theorem*. It is referred as the only one, among the five conditions of that theorem, that can be questioned without provoking an evident modification of the subject's essence. The rejection of Condition #3 would allow the inclusion, in the set of possible decisions, of hypothetical alternatives specifically selected with the objective of evaluating the preference intensity over the truly viable options. However, if the decision makers become aware that the hypothetical alternatives are impracticable, they may deliberate on their authentic propensities and whence make their "moves" as if they were part of a strategic game. Then, the outcomes may be distorted in relation to the original goal, which is the search for an aggregated expression of the collective preference, or social satisfaction function (Welfare

◇ Axiom 5: Transitivity of Preference and Indifference Relations

This axiom is an extension of Axiom 1, generalizing the transitivity concept to any number of prizes A_i . Several studies, as those performed by Allais [ALLA53] and Tversky [TVER69], [TVER74] have shown that the condition of preferences transitivity cannot be always taken for granted when making decisions under risk. Furthermore, there are some situations where an individual simply will not be able to rank the alternatives presented to it²⁰.

◇ Axiom 6: Monotonicity

A lottery $L_1: [pA_1, (1-p)A_r]$ is preferred or indifferent to another lottery $L_2: [p'A_1, (1-p')A_r]$ if and only if $p \geq p'$, given that $A_1 \approx \succ A_2 \approx \succ \dots \approx \succ A_r$.

Although Utility Theory is a successful model to explain decision making under risk, there are controversies about generalizing it to all circumstances of this kind. An important research on this matter is the *Prospect Theory*, that is discussed in the sequence.

2.6.2 - Prospect Theory

One of the postulates to the application of the expected utility method is the *risk aversion*²¹, which asserts the concavity of the utility function, that is, $u'' < 0$. Risk aversion can be characterized by the preference to a certain positive gain A in opposition to a lottery with an expected gain $E(A)$. In the expected utility approach the utilities are weighted by their respective probabilities of occurrence. Kahnemann and Tversky [KAHN79] have made a criticism of that method by showing several empirical results which demonstrate its

Function). The obtainance of that function was proved impossible under the five conditions of Arrow's Theorem [ARRO51].

²⁰ A startling example of that situation is in a game discovered by the mathematician Walter Penney, presented by Martin Gardner [GARD74].

²¹ The other two are (1) Expectancy: $U(A_1p_1; \dots; A_rp_r) = p_1u(A_1) + \dots + p_ru(A_r)$, and (2) Aggregation of utilities: a lottery $(A_1p_1; \dots; A_rp_r)$ is said to be *acceptable* from a previous ownership of an asset w if $U(w + A_1p_1; \dots; A_rp_r) > u(w)$. The debate about risk aversion presented in Prospect Theory has been cited in many other more recent works, like for example [LINV91] and [JOSE92].

transgression in various instances. They have proposed, as a substitute to the former, what they called *prospect theory*.

Prospect Theory focuses on some behavioral features called *effects*, which are associated to the transgression of the expected utility principle. Those are:

- *Certainty effect*: When a positive gain tends to 100% of certainty, it is generally preferred over other risky options, even if they can yield a higher prize, within some variable limit. An experiment composed of two problems, each with two lotteries. Problem 1 had lottery A:(4000, 0.80; 0, 0.20)²² and lottery B:(3000, 1). In problem 2 consisted of the lottery C:(4000, 0.20; 0, 0.80), and D:(3000, 0.75; 0, 0.25). Over 80% of the 95 individuals consulted in the poll preferred B in problem 1, but in problem 2, 65% selected C. Those results contradict the criterion of selecting the alternative that yields $\max E(U)$. An explanation provided by Kahnemann and Tversky for this behavior lies in the attractiveness that the *certainty* of the prize \$3000 exerts on the decision.
- *Reflection effect*: If instead of gains the choice regards losses, there is an inversion of the certainty effect, and the majority of individuals usually opt for “gambling”, in despite of a likely greater loss, also within some variable limit.
- *Isolation effect*: Consists in a tendency to make decisions based only in parts of a compounded lottery. For example, consider a lottery with two stages: L_1 , with 75% of chance of ending the game with zero gains, and 25% of proceeding to the second stage, where two sub-lotteries are available, $L_{2,1}$:(4000, 0.80; 0, 0.20) and $L_{2,2}$:(3000, 1). The behavior of 141 individuals was investigated, and 78% of the sample opted for $L_{2,2}$. This result is opposed to the conclusion obtained in the choice problem regarding the certainty effect, which nevertheless is the same as the present after the combining the probabilities of the two stages.

Kahnemann and Tversky suggest that the variable that mostly affects the valuation of the Utility Function is the *change* that a risky prospect may impose to a pre-existent amount of wealth. They conjecture that this factor surpasses the importance of the final position to be achieved through the game.

²² The notation (4000, 0.80; 0, 0.20) means a lottery that yields \$4000 with a probability of 80% and zero with a 20% chance.

Prospect Theory has been developed with the objective of explaining the observed violations of Utility Theory. The model distinguishes two basic sequential phases in the decision making process. The first is called the Edition Phase, which aims at organizing the available options and comprises the operations *coding*, *combination*, *segregation*, *canceling*, *simplification* and *dominance detection*. Those procedures attempt to make more explicit to a player the options represented by the lotteries, without altering their respective relations between gains and losses. For instance, the combination operation would transform a lottery $L:(10, 0.3; 10, 0.4)$ in $L':(10, 0.7)$.

The second phase is the *evaluation*, whose purpose is to establish an attractiveness measure for each previously “edited” lottery²³. The measure employed in the Prospect Theory is two-dimensional and utilizes the scales $(\pi)^{24}$ and $v(a_i)^{25}$, that are subjective evaluations of the probabilities and prizes.

A lottery is denominated *regular* if $(p+q) < 1$, or $a_1 \geq 0 \geq a_2$, or if $a_1 \leq 0 \leq a_2$. If it is regular, the basic equation of Prospect Theory follows, which establishes the final value V of the lottery as a function of $\pi(\cdot)$ and $v(\cdot)$.

$$V(a_1, p; a_2, q) = \pi(p) \times v(a_1) + \pi(q) \times v(a_2) \quad \text{Eq. 2.15}$$

The appearance of function $v(a_i)$ is a concave line for positive values of a_i (gains) and a convex line for negative values (losses). Its gradient also decreases as the absolute value of a_i increases.

The function $\pi(p)$ is *not* a probability measure, but instead a measure of *how* the decision maker *perceives* the probabilities, being called “decision weight”. To illustrate the difference between a probability value and $\pi(p)$, consider the game of tossing a fair coin, depicted by $(\$1000, 0.50; \$0, 0.50)$. Empirical evidence hint that in this case $\pi(0.50) < 0.50$. If a gambler is offered an option of picking a sure prize instead of tossing the coin, it would eventually settle around \$350, which means that $\pi(0.50) \cong 0.35$. $\pi(p)$ is

²³ The formulation presented by Kahnemann and Tversky refers to simple lotteries of the type $(a_1, p; a_2, q; 0, (1-p-q))$, where $p+q \leq 1$.

²⁴ $\pi(p)$ does not obey the axioms of Probability Theory. In general, $\pi(p) + \pi(1-p) < 1$.

²⁵ In that model, the prizes a_i are the deviations of the absolute prizes A_i in relation to an arbitrated reference point A^* .

monotonically increasing in its domain $[0, 1]$ as well as its gradient, indicating that small probabilities (near zero) tend to be overvalued, the inverse occurring with high probabilities, that are usually undervalued²⁶. However, it has been observed that $\pi(p)$ presents an erratic behavior for very small or very high values of p , contradicting the pointed property in the sense of either a neglecting small probabilities or attributing certainty to very low or very high values of p , respectively.

2.6.3 - Rationality and the Maximization of Expected Utility

As seen, the determination of a quantitative measure of utility is accomplished through the observation of choices by a given agent in risky situations. This agent is expected to be acting in accordance to the principle of rationality that dictates the maximization of expected utilities in every circumstance. This assumption entails the rise of a tautology, because expected utility *is defined* as that which is maximized [RAPO90].

The requirement concerning the consistency of choices among lotteries for the determination of an agent's utility function is also another problematic factor, because it is not always endorsed by observed evidence. Therefore, the association between irrationality and inconsistency is intimately related to the method employed just described for the determination of the cardinal scale of utility.

Maurice Allais posed important questionings to the expected utility hypothesis. According to his view, the historical development of the theoretical concept of the pure psychology of risk has passed through four successive stages [ALLA79]. Given a random prospect, composed of various amounts of gains, its value V is represented by the following equations in each of the considered stages:

- Stage one: Mathematical expectation of gains expressed in monetary units

$$V = \sum_{i=1}^n p_i g_i, \quad \text{Eq. 2.16}$$

where g_i are monetary gains and p_i are objective probabilities of occurrence associated to each g_i .

²⁶ That phenomenon may partially explain why people bet in lotteries or make insurance.

- Stage two: Substitution of psychological values (*utilities*) for the monetary units in Equation 2.16.

$$u(V) = \sum_{i=1}^n p_i u(g_i) \quad \text{Eq. 2.17}$$

where $u(g_i)$ and $u(V)$ are the utilities, or psychological values, of the alternatives and of the combined prospect, respectively. This approach was originally developed by Bernoulli, in order to expose his St. Petersburg paradox.

- Stage three: Substitution of *subjective* for *objective* probabilities in Equation 2.17.

$$\bar{u}(V) = \sum_{i=1}^n \bar{p}_i u(g_i), \quad \text{Eq. 2.18}$$

where \bar{p}_i are the subjective probabilities of occurrence of each of the outcomes g_i and $\bar{u}(V)$ is called the neo-Bernoullian formulation of the expected utility and was also employed by Neumann and Morgenstern in their work on the Theory of Games.

- Stage four: Inclusion of the dispersion of the utilities, taking into account the probability distribution of their psychological values around the mean.

$$\bar{u}(V) = f(\Psi(\gamma)) \quad \text{Eq. 2.19}$$

where $\gamma = u(g)$, $\psi(\gamma)$ is the probability distribution function of γ , f is a functional of $\psi(\gamma)$ and $\bar{u}(V)$ is the revised expected utility.

Allais supported the view contained in stage four because he sustained that the neo-Bernoullian method (stage three) could give margin to inconsistencies. The demonstration of this fact is illustrated by Allais' classic example of choice involving risk, as follows. Consider the two independent situations 1 and 2, each with two choices A and B:

| | |
|----------|--|
| Choice A | \$10 000 with 100% certainty |
| Choice B | \$50 000 with probability 10% , or \$10 000 with probability 89% , or \$0 with probability 1%. |

Table 2.2 - Allais' Situation 1

| | |
|----------|--|
| Choice C | \$10 000 with probability 11% , or \$0 with probability 89% |
| Choice D | \$50 000 with probability 10% , or \$0 with probability 90% |

Table 2.3 - Allais' Situation 2

When confronted to the choices depicted in tables 2.2 and 2.3, many persons prefer A to B in situation 1 and D to C in situation 2. But, if the agents are rational, that should not happen, because this pattern of preferences is inconsistent with the principle of choice in accordance to the maximum expected utility. The inconsistency is independent of the utility attributed by the agents to the monetary amounts²⁷. Nevertheless, it cannot be said that people who choose in this way are irrational. They are only not using the criteria of maximizing the expected utility in their decisions, what is quite different. Thus, this principle is not appropriated to define rationality.

One of the basic premises of the traditional approach is that each player will always be able to select the alternative that maximizes the expected value of its respective utility function, among the options involving risk or uncertainty. This condition refers to the concept of *rationality* of the players, which is a very strong assumption with broad implications. Nevertheless, if this condition is satisfied, how should each participant decide, taking into account that the others' actions will also be *rational*?

²⁷ For an explanation of the details of the inconsistency, usually known as the “Allais Paradox”, see [RAPO90], p.96.

For example, suppose a game played by two contestants, P_1 and P_2 . The sets $X=\{x_1, x_2, \dots, x_m\}$ and $Y=\{y_1, y_2, \dots, y_n\}$ represent their available actions. The convenience of P_1 to adopt a particular x_i depends on which y_j P_2 will choose, and conversely. However, being P_1 and P_2 both rational, the situation is symmetric. This reasoning leads to the conclusion that a rational choice depends on itself, thereby constituting a looping that does not explain the process [BACH87]. On the other hand, some situations may arise where there are mutually preferred alternatives, expressed by the utilities $U_1(x_k | y_1)=\max$ and $U_2(y_1 | x_k)=\max$. The pair of actions (x_k, y_1) are said to be a *Nash equilibrium*²⁸.

2.6.4 - Other Criticisms to the Maximum Expected Utility Method

An interesting aspect of decision making regards the *role* that an individual assumes during that task. In an experiment performed by Birnbaum et al. [BIRN92], some volunteers were separated in three distinct groups with the role models of *buyers*, *sellers* and *umpires*. Depending on its respective characterization, each group showed diverse preference profiles even when the same individual was confronted with the same decision problem. Birnbaum's experiment consisted in asking each individual to attach a "just" price to lotteries, respectively assuming one of the role models at a time. The assignment of values, which indicated the preferences, varied according to the assumed role. For example, the lottery (8, 0.5; 80, 0.5) had a higher value than the lottery (32, 0.5; 40, 0.5) under the *seller's* point-of-view. However, that appraisal was inverted when the decision regarded *buyers*.

Exploring another angle of the non-general applicability of Utility, Gul [GUL91] introduced a model that he called *Theory of Disappointment Aversion*. The theory has equally the same objective as the previous methods, that is to supply a better explanation of how rational agents make decisions under risk, and it has three basic features:

- a. It Includes the traditional Expected Utility method as a particular case;
- b. It is consistent with Allais Paradoxes²⁹;
- c. It is the most restrictive model that satisfies (a) and (b) conjointly.

²⁸ For the definition of Nash and other equilibrium concepts, see section 2.9.

²⁹ In his paper Gul presents an example of preference inconsistency owed named *Allais Ratio Paradox* - op. cit [ALLA79].

In Gul's approach, the individual preferences are represented by a parameter u (utility function) and a real number b , $b > 1$, which is associated to attitudes toward risk³⁰.

2.7 - Extensive and Normal Forms of a Game

2.7.1 - Extensive Form

A game in its *extensive form* is characterized by the following set of rules:

- i. Representation by means of a finite graph tree, with nodes standing for decision points and branches indicating the available alternatives departing from each node;
- ii. Partition of the set of all nodes in $n+1$ subsets, n of them corresponding to the players nodes, plus another set referring to moves made by any random process;
- iii. Existence of a probability distribution associated to every node where the move is performed by the random process, thereby establishing the chance of each particular branch being selected;
- iv. Grouping of the decision points of each player in *information sets*, indistinguishable one from another in terms of the number of branches that diverge from each of the their nodes, due to incomplete information;
- v. Equivalent identification of the branches that depart from the same node, so to avoid the categorizing of the alternatives;
- vi. For every player there exists a linear utility function defined over the terminal nodes of the graph tree.

Conditions i-vi aim not only at defining the rules of the game, but also have the objective of designating the preference patterns of the players through the mentioned utility function. In this manner, distinct preferences manifested by any player over the outcomes of a game imply in a different game regarding its extensive form. The main motivation for defining a game in its extensive form is checking the existence of an equilibrium point for pure strategies.

³⁰ The variable b in Gul's model is similar to the function $\pi(p)$ in Kahnemann and Tversky's Prospect Theory, though the former is presented in a more mathematically formal and elaborated context.

2.7.2 - Normal Form

A game in its normal form consists of the reduction of the extensive form into only one move, expressed by the selection, by every one of the n participants, of an available pure strategy. The utility U_i for a player i that yields from a set of pure strategies s_i employed by the n players can be defined by

$$U_i(s_1, s_2, \dots, s_n) = U_i(\alpha), \quad \text{Eq. 2.20}$$

where α is the collection of moves represented by the pure strategies picked by the participants. If the choice of strategies is not deterministic, then it can be assumed that a specific match α of the game will happen with probability p . In that case the utility U_i is described by Equation 2.21.

$$U_i(s_1, s_2, \dots, s_n) = \sum_{\alpha} p(\alpha) \cdot U_i(\alpha). \quad \text{Eq. 2.21}$$

In that way, the function $U_i(\cdot)$ altogether with the set of strategies $\{s_i\}$, $i=1, 2, \dots, n$, determine for each player i its gain in the match of a game. This simplification, that can be applied to any game, is called the normal form. That concept can be extended to the hypothesis of mixed strategies. The gain of a player is then defined by the value of its Expect Utility — albeit the controversy³¹ — and for the probability distributions of each player's pure strategies.

Besides the extensive and normal forms, a game can still be modeled according to its characteristic function, not reviewed in this dissertation³².

2.8 - Competitive Non-zero-sum Games

When two players P_1 and P_2 participate of a *strictly competitive game*, given its zero-sum attribute, the benefit that each one obtains is always accomplished by imposing a loss to the other. In other words, any preference about the outcomes will be the inverse for each other. When this condition does not verify, a game is no more strictly competitive, and the utility functions of the players do not add up to zero.

³¹ See section 2.6.

³² For an explanation of the Characteristic Function, see op. cit [LUC57] and [BLAC54].

Several differences exist between the analysis of zero-sum and non-zero-sum games. A first aspect to be mentioned regards the significance of the information derived from divulging or knowing previously which strategy a player will adopt. In strictly competitive games, it has been seen that the participants, when acting rationally, should not announce which particular strategy they would implement, because that contained valuable information, which could be used by an opponent to secure more advantageous positions and greater gains in the match. As a consequence of the zero-sum feature, the player's benefit would decrease. However, this restriction is not applicable in non-zero-sum games. It is even feasible that the anticipated disclosure of a player's intended may imply in mutual gains, if the revelation is perceived by the other as authentic, inducing it to take a compatible action, and if the strategy is actually fulfilled. It must be noted that revealing the strategy should not equal a *negotiation process*, which only takes place in cooperative games³³. The best known non-zero-sum competitive game is the *Prisoner's Dilemma*, that is reviewed in greater detail in Chapter 3 of this dissertation.

Another significant contrast between zero-sum and non-zero-sum games regards alterations in the concept of strategy equilibrium, that is reviewed subsequently.

2.9 - Dominant and Nash Equilibria

Consider a game with the following payoff matrix:

| | | Player B | |
|----------|------------|-----------|-----------|
| | | β_1 | β_2 |
| Player A | α_1 | (10, 4) | (0,0) |
| | α_2 | (0, 0) | (5, 8) |

Figure 2.2 - Payoff Matrix of a Non-Strictly Competitive Game

The strategy pair (α_1, β_1) and (α_2, β_2) are in *equilibrium*, because α_1 and α_2 are the best moves against β_1 and β_2 , respectively. On the other hand, neither (α_1, β_2) nor (α_2, β_1)

³³ See section 2.10.

are in equilibrium, and those pairs do not yield equivalent payoffs to the players. In zero-sum games, if two pairs of strategies (α_1, β_1) and (α_2, β_2) are in equilibrium, so are (α_1, β_2) and (α_2, β_1) ³⁴.

If the players decide to employ mixed strategies, in order to obtain their respective security levels, those would be the **maxmin** strategies, namely the combinations $(\frac{1}{3}\alpha_1, \frac{2}{3}\alpha_2)$ for player A and $(\frac{2}{3}\beta_1, \frac{1}{3}\beta_2)$ for player B. In that case, the security levels are $10/3$ and $8/3$. It must be noted that the maxmin strategies do *not* constitute an equilibrium, so those decisions are not the best reply against each other's feasible actions. For example, assuming that the player A deduces that B will use its maxmin strategy, A should play β_2 , whence attaining an expected gain of $16/3$. If A understands that reasoning, it should opt for α_2 . But A can also guess that B is going to use minimax, and in this case A must play α_1 , which yields an expected gain of $20/3$. It can be seen that a reasoning loop arises, and this is an inherent difficulty of non-cooperative, non-zero-sum games. If the game were iterated and cooperative, in the sense that a pre-play negotiation is allowed, then the players could reach an agreement where the pairs (α_1, β_1) , (α_2, β_2) would be used a certain proportion of matches each.³⁵

2.9.1 - Dominant Strategy Equilibrium

A strategy s_i^* is a *dominant strategy* if it is a player's strictly best response to any strategies the other players might adopt. This means that s_i^* yields it the greatest payoff in any circumstances of the game [RASM89]. Otherwise, a strategy is called *dominated*.

$$u_i(s_i^*, s_j) > u_i(s_i', s_j), \forall j \neq i, \forall s_i' \neq s_i^* \quad \text{Eq. 2.22}$$

When all players employ their respective dominant strategies, the resulting combination is denoted a *dominant equilibrium*.

If s_i^* is no smaller than any s_j , $\forall j \neq i$, and greater for some j 's, then it is denominated a *weakly dominant strategy*. An *iterated dominant strategy equilibrium* is

³⁴ Furthermore, in zero-sum games the utilities associated to any pair of pure strategies are the same.

³⁵ The proportion could be arbitrated based on the criterion of equalizing the accumulated utilities. In the example, that proportion is $\frac{1}{3}$ for (α_1, β_1) . This is a very naive solution, which presupposes linear properties for the utilities and neglects the eventual distinct bargaining power of the players.

found by excluding a weakly dominated strategy from one of the players, recalculating to check the remaining weakly dominated strategies of all players and repeating the process until only one strategy is left for every participant.

2.9.2 - Nash Equilibrium

In 1951, the game theorist John Nash proved that any 2-person non-cooperative game with a finite number of strategies has at least one pair of mixed strategies in equilibrium³⁶. The majority of games do not have even an iterated dominant equilibrium, but they may possess a Nash equilibrium, which is the "... most important and widespread equilibrium concept" [RASM89].

A strategy combination s^* is a Nash equilibrium if no player has incentive to deviate from its choice provided that the opponent maintains its position. Every dominant strategy is a Nash equilibrium, but the inverse is not necessarily true.

Given two equilibrium pairs, this does not necessarily mean that the expected gains of each player are equal. However, if it is the case, that is, $E(u_\alpha(\alpha_1, \beta_1)) = E(u_\beta(\alpha_2, \beta_2))$ and $E(u_\alpha(\alpha_1, \beta_2)) = E(u_\beta(\alpha_2, \beta_1))$, the pairs (α_1, β_1) , (α_2, β_2) are said to be *equivalent*, and also called *interchangeable* if (α_1, β_2) and (α_2, β_1) are in equilibrium³⁷.

The proof of the existence of at least one equilibrium pair of mixed strategies in non-cooperative games is associated to the concept of *solution* of a game. In Nash's sense, a solution exists if any equilibrium pair is interchangeable. Thus, a game's solution is the set of all strategies that obey the latter condition, which must not be equivalent.

For clarity, consider the game³⁸ represented by the payoff matrix of Figure 2.3.

³⁶ Partly due to this work, Nash received the 1994 Nobel prize in Economics, altogether with J. Harsanyi and R. Selten.

³⁷ From that property, in a strictly competitive game, each pair of equilibrium strategies is both equivalent and interchangeable.

³⁸ The example was obtained from [LUC57]. That game has an interesting (and frustrating) feature. Each player's gain is completely determined by the other's choice, therefore implying in a total indifference regarding a participant's own move. Observe that the pure strategy pair (a_2, b_2) would be the obvious pick if a previous contract between the players could be sealed.

| | | Player B | |
|----------|------------|-----------|-----------|
| | | β_1 | β_2 |
| Player A | α_1 | (2, 5) | (4, 5) |
| | α_2 | (2, 6) | (4, 6) |

Figure 2.3 - Payoff Matrix of a Non-Strictly Competitive Game

It can be seen in this game that all possible pairs of pure and mixed strategies are in equilibrium. Hence, the game has a solution in Nash’s sense³⁹. However, the expected gain for two different pairs of strategies are not equal. The resulting payoffs, depending on the strategies employed are 4 (α_2, β_2) or 2 (α_1, β_1), for player A and 6 or 5, for player B. It is clear that both players would rather have the outcome (α_2, β_2) than (α_1, β_1). In that case it is said that the former pair of strategies *conjointly dominates* the latter, which is called *conjointly inadmissible*. On the other hand, if there exists another combination of strategies that is not conjointly dominated by any other pair, then it is named *conjointly admissible*, and only in this case.

If among the conjointly admissible strategies of a non-cooperative game there is at least one equilibrium pair, and if all those strategies are simultaneously equivalent and interchangeable, then the game is said to have a solution in the *strict sense*. In this way, the concepts of solution in Nash’s and in the strict sense differ. A non-cooperative game may be solvable according to one interpretation, but not to the other. Let us consider another example, shown in Figure 2.4.

³⁹ The literature often mentions the games *Battle of the Sexes*, *Boxed Pigs* and *Pure Coordination* as usual examples of the application of the Nash’s equilibrium concept.

| | | Player B | |
|----------|------------|-----------|-----------|
| | | β_1 | β_2 |
| Player A | α_1 | (1, 2) | (-1, -1) |
| | α_2 | (0, 0) | (3, 4) |

Figure 2.4 - Payoff Matrix of a Strictly Solvable Competitive Game

The strategies (α_1, β_1) and (α_2, β_2) are in equilibrium, but they are not interchangeable since $u_{A, B}(\alpha_1, \beta_2) \neq u_{A, B}(\alpha_2, \beta_1)$ and thus no solution in the Nash's sense. Nonetheless, the game has a solution in the strict sense because (α_2, β_2) conjointly dominates (α_1, β_1) .

2.10 - Cooperative Games

According to the postulates of Game Theory, *Cooperative Games* are those where the players may communicate before the match begins. This interaction has the objective of allowing the accomplishment of some kind of agreement regarding the strategies that each contestant will implement in the game.

That relaxation of the foregoing restriction present in competitive games (either strictly or not) adds several new factors to the analysis, therefore increasing their complexity. The first aspect refers to the influence that the number of participants exerts in the investigation. While in competitive games the players are isolated from each other, and can rely only in their respective reasoning and intuition to decide about their moves⁴⁰, cooperative games admit the possibility of forming coalitions with many structures⁴¹. Other aspects regard the agreements' characteristics, including their detailing levels, eventual ambiguities, and mainly, its enforceability⁴².

⁴⁰ In repeated competitive games, the sequence of outcomes from past iterations can be seen as a kind of information exchange, even though there is no explicit negotiation among the players.

⁴¹ The way the structures are formed may be affected by the allowance for *threats* or future concession of advantages to the opponent(s) (*side payments*). Furthermore, the coalition formed for the first match may not endure along the rest of the game, due to changes in interests and performances achieved.

⁴² In "*The Prince*" Maquiavel suggests that a wise sovereign does not have to abide to his or her word, when doing that is contrary to the prince's own interest. In "*Maximes pour le Dauphin*" Louis XIV recommends that in every Treaty, a clause that can be easily violated by the other side should be included, so that the whole agreement can be turned void, whenever such act is in the State's interest [SFOR53].

Bargaining power can be derived from different utility functions, from the access to relevant information, from the capacity to absorb temporary losses and from the capacity of formulating credible threats, among other factors. Bargaining power has a great influence in the development and conclusion of a cooperative game, especially when the game is repeated and has no deterministic defined end known by the parties involved. That point-of-view is strengthened if one understands that in social relations, be they between individuals, between companies or even nations, the consequences of the repeated character of the transactions will play their roles in the participants decisions.

This repetition, even if occurs in non-identical circumstances, has the dynamics of a very complex supergame, where the agents, interests, conjuncture, coalitions and even rules are ever-changing. Under this view, it may be assumed that some type of cooperation will always be present in the game.

It is of fundamental importance to stress that the meaning of *cooperation*, in that approach, totally departs from the usual definition, and on that account it must not be interpreted as a synonym of altruism, justice, good will or other qualities induced by the players' wish to self-renounce to prospective advantages that may be obtained by independent actions⁴³.

In accordance to the model developed in this dissertation, the focus of the research is centered in games involving pairs of players, although the population that takes part in the interactions is composed of multiple players⁴⁴.

2.10.1 - Two-Person Cooperative Games

In this section, the following conditions are assumed:

- i. Communication between players is total and unambiguous, and each participant learns what is stated from the other without any distortions or misunderstandings;

⁴³ As a matter of fact, it seems too rigid to classify the games in cooperative or not. In most practical situations, the communication between the players can assume various degrees, ranging from nonexistent to total.

⁴⁴ Chapter 5 consists of a investigative model of the Fuzzy Iterated Prisoner's Dilemma with a population of 515 distinct types of players, and in Chapter 6 presents an application of an one-sided Fuzzy PD to a market share simulation, involving 1000 buyers and variable number of sellers (Firms).

- ii. The games have fixed rules, which are mandatory, particularly regarding the obedience of agreed contracts;
- iii. The preferences about the feasible outcomes are not susceptible to alterations due to the course of negotiations;

In some real world situations, in spite of condition (iii), it can happen that a participant might refuse to take part in the pre-play negotiations on account of strategic reasons. The explanation for such behavior is the avoidance of getting exposed to intimidation that may possibly come from an adversary. *Plausible* threats, which are allowed in cooperative games, refer to the announcement of retaliations in case some player declares that it will play in an unsatisfactory way, under the point-of-view of the opponent that instated the threat.

But if there is actually a potential threat in perspective, why should anyone avoid being *explicitly and directly informed* of it? The principal motive, apart from other moral considerations, lies in the fact that a threat becomes *common knowledge* upon being stated and understood, and that event exerts a significant influence in the players' decisions⁴⁵.

2.10.2 - Von Neumann and Morgenstern Solution

| | | Player B | |
|----------|------------|-----------|-----------|
| | | β_1 | β_2 |
| Player A | α_1 | (2, 1) | (-1, -1) |
| | α_2 | (-1, -1) | (1, 2) |

Figure 2.5 - Payoff Matrix of a Cooperative Game

The solution proposed by those authors for cooperative games is based on the determination of what is called the *negotiation set*. That set excludes all points that represent pairs of payoffs relative to conjointly dominated strategies, and also other points

⁴⁵ In Martin Gardner's *Mathematical Games* column once presented in *Scientific American* [GARD78], an example involving a logic game clearly and simply demonstrates how the existence or not of common knowledge among the participants can dramatically change the outcomes, leading to an apparent paradox if that aspect is neglected.

which imply in the achievement, by any player, in gains that are inferior to those that could be obtained without any agreement. For clarity, let us reflect on the payoff matrix depicted in Figure 2.5.

The cells in the secondary diagonal are obviously undesired by both players. In a repeated game, a plausible sequence of moves would be alternating, in phase, of the strategies (α_1, β_1) and (α_2, β_2) , what would yield for each participant an average gain of $\frac{3}{2}$. But if the game is played only once, this value is not supposed to be achieved without a previous stage where an arrangement between the players is agreed. In the hypothesis a negotiation betides, (α_1, β_1) and (α_2, β_2) could obtain from a random device yielding a 50%-50% probability for each pair.

The negotiation set for this game is illustrated if Figure 2.6

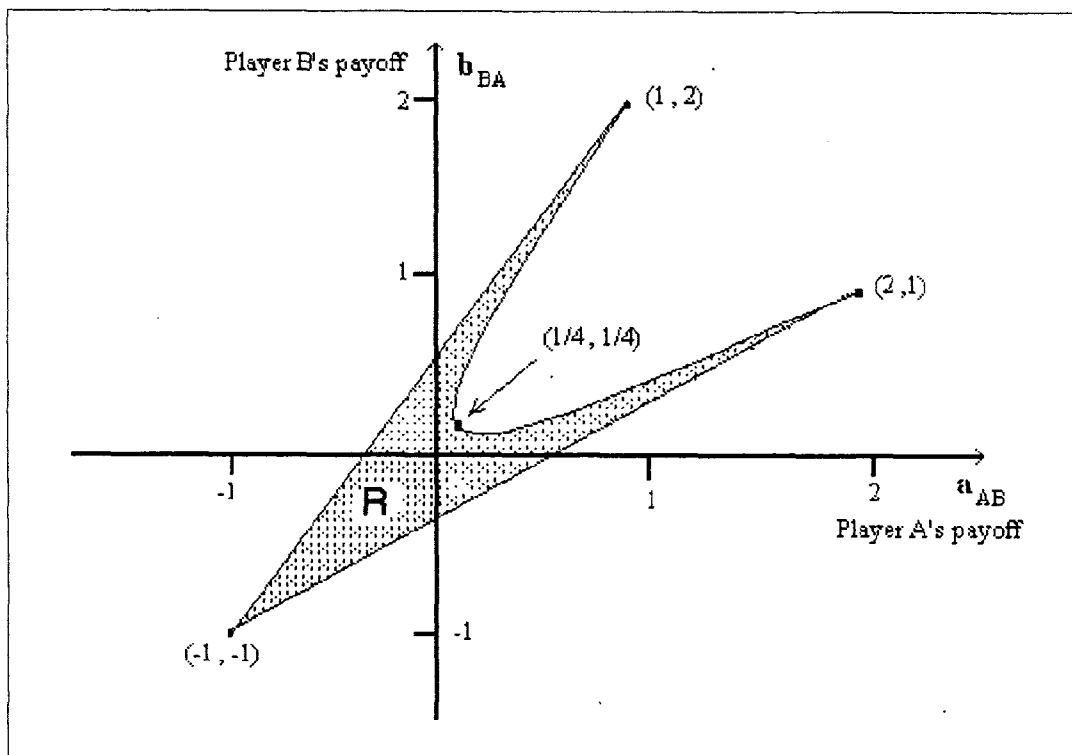


Figure 2.6 - Negotiation Set R for the Von Neumann and Morgenstern Solution of a Cooperative Game with Two Independent Mixed Strategies.

The shaded area **R** contains all the points which embody the pairs of payoffs (a_{AB}, b_{BA}) defined by the adoption of a pair of mixed strategies (α_A, β_B) , where:

$$\alpha_A = \{[(p_A \alpha_1, (1-p_A) \alpha_2)], p_A \in [0, 1]\} \quad \text{Eq. 2.23(a)}$$

$$\beta_B = \{[p_B \beta_1, (1-p_B) \beta_2]\}, p_B \in [0, 1]. \quad \text{Eq. 2.23(b)}$$

The correspondence between any pair (α_A, β_B) and (a_{AB}, b_{BA}) , and vice-versa, is unique.

The point that represents the payoffs $(\frac{3}{2}, \frac{3}{2})$ does not belong to \mathbf{R} . That average gain cannot be reached when the players decide on the values of p_A and p_B to be used in their respective mixed strategies (α_A, β_B) independently, as in a competitive game. Conversely, if the game is cooperative, then it is possible to the players to assent on a pact which product is a *conjoint mixed strategy*. That scheme incorporates a specific probability distribution over all the possible pairs of *pure conjoint strategies* (α_i, β_j) , $i=1, 2; j=1, 2$.

Regarding the example, a common payoff of $\frac{3}{2}$ is brought about by the conjoint mixed strategy designated by $[\frac{1}{2}(\alpha_1, \beta_1), 0(\alpha_1, \beta_2), (\alpha_2, \beta_1), \frac{1}{2}(\alpha_2, \beta_2)] \Leftrightarrow (\frac{3}{2}, \frac{3}{2})$.

Designating a generic conjoint mixed strategy by $\omega_{AB}=(\alpha_A, \beta_B)$, and by Ω the set that contains all feasible ω_{AB} , the expected gains (utilities) associated to Ω are

$$u_A(\Omega) = \sum_{A, B} a_{AB} = \sum_{AB} u_A(p_{AB} \cdot \omega_{AB}), \omega_{AB} \in \Omega, p_{AB} \in [0, 1]. \quad \text{Eq. 2.24(a)}$$

$$u_B(\Omega) = \sum_{A, B} b_{AB} = \sum_{AB} u_B(p_{AB} \cdot \omega_{AB}), \omega_{AB} \in \Omega, p_{AB} \in [0, 1], \quad \text{Eq. 2.24(b)}$$

where a_{AB}, b_{AB} are the utilities that derive from a particular conjoint mixed strategy ω_{AB} and p_{AB} is the probability of electing a particular ω_{AB} .

Noting that ω_{AB} is an argument of the utility functions u_A and u_B , the pairs (a_{AB}, b_{AB}) that derive from each ω_{AB} can be mapped in a convex region R' configured by the points p , q and r shown in Figure 2.7.

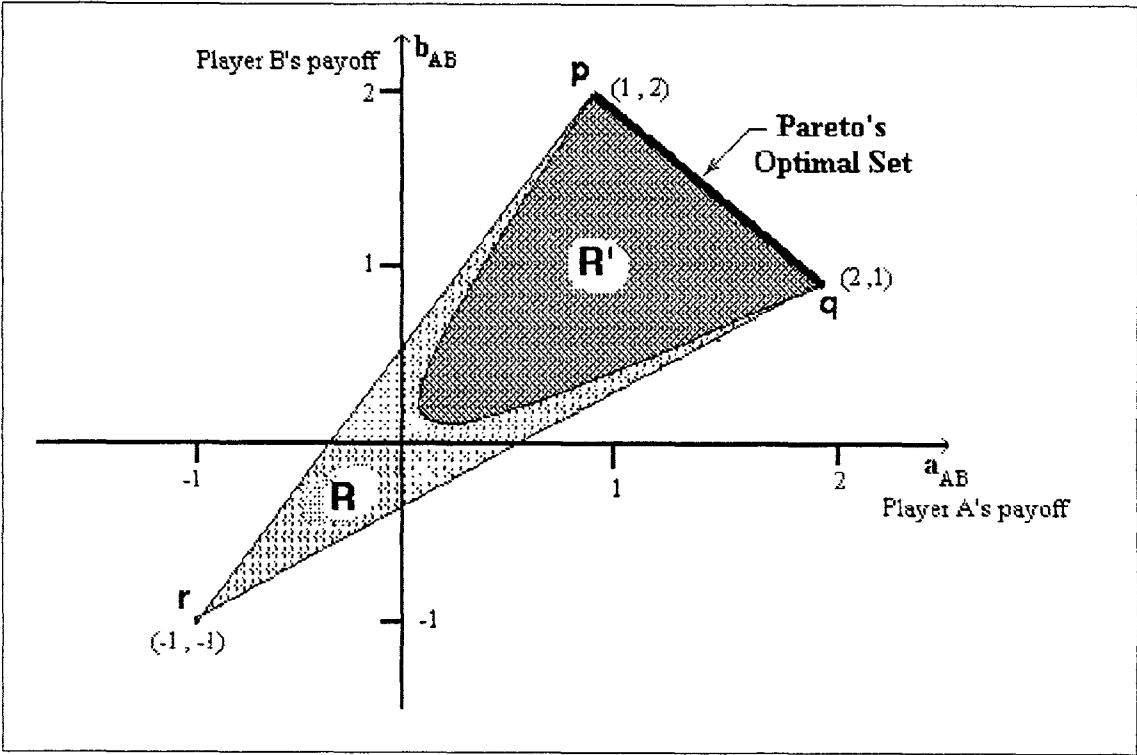


Figure 2.7 - Cooperative Solution Set R' and Pareto's Optimal Set

A point $(a'_{AB}, b'_{AB}) \in R'$ is said to be *conjointly dominated* by another point $(a''_{AB}, b''_{AB}) \in R'$ if $a''_{AB} \geq a'_{AB}$ and $b''_{AB} \geq b'_{AB}$ simultaneously. Having in mind that R' stands for the players' utility values or expected gains, the participants, being rational, will seek the maximization of their respective payoffs. Hence, all pairs (a_{AB}, b_{AB}) that are conjointly dominated shall be in due course discarded by them. The pre-play negotiations will then close in upon the selection of conjoint mixed strategies which outcomes only include the points that belong to R' that do not have the former characteristic. Those points correspond to the line segment depicted by pq in Figure 2.7, which is called *Pareto's Optimal Set*. When the players agree in restricting their strategies to those associated to pq , the possibility of acquiring higher payoffs by means of further negotiations is exhausted, because beyond that settlement the preferences become strictly opposed. In the example,

player A prefers q and B prefers p , that correspond to the conjoint mixed strategies $\omega'_{AB} = [1(\alpha_1, \beta_1), 0(\alpha_2, \beta_2)]$ and $\omega''_{AB} = [0(\alpha_1, \beta_1), 1(\alpha_2, \beta_2)]$, respectively.

However, not all points from the Pareto's optimal set are achievable, in terms of realistic plausible expectations. For instance, consider a game whose region R' is bounded by the polygon $abcd$ in Figure 2.8.

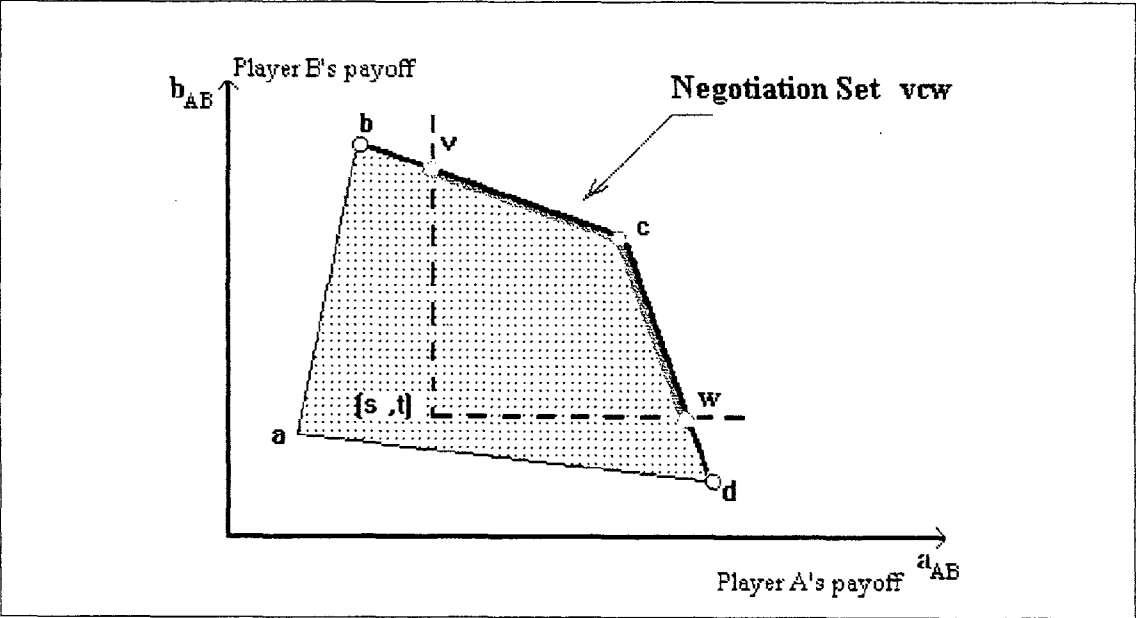


Figure 2.8 - Pareto's Optimal Set and the Negotiation Set

In Figure 2.8, Pareto's optimal set is defined by the line bcd . Supposing that the point (s, t) stands for the payoffs when the players implement their maxmin strategies, in the non-cooperative version of a game, it is out of question that none of them would accept a deal that would yield an inferior benefit. Accordingly, Von Neumann and Morgenstern's solution space for the cooperative game of Figure 2.8 is the subset vdw from Pareto's set, denominated Negotiation Set (N).

The subset N is often viewed as the limit, in terms of mathematical treatment, for the imposition of restraints that could shrink the solution space for a cooperative game. The selection of a particular point in N would then depend in other extrinsic attributes⁴⁶.

⁴⁶ Some extraneous factors regard the players' psychological aspects, their bargaining power and conjuncture variables.

In the example, the symmetry of the payoff matrix hints that the proposed cooperative solution $(\frac{3}{2}, \frac{3}{2})$ is quite reasonable. Nevertheless, it seems that idealizing a perfect, equitable settlement is seldom within reach in realistic situations. This is so because each player's role⁴⁷ greatly affects a solution that might be collectively accepted as *just*. Therefore, the asymmetry in games may be present not only in the payoff matrix, but also in the players' capacity to achieve successful compromises, in the resources to make credible threats, etc.

What can be deduced from those arguments is that the concept of solution to be found by the participants of the game themselves is not satisfactory, even when the solution is restricted to **N**. Therefore, a need to the introduction of a third party's *arbitration scheme* arises, whose goal is to establish an impartial arrangement.

2.10.3 - Arbitration Schema⁴⁸

An arbitration scheme⁴⁹ is a criteria for settling a conflict of interest, where a third party, not directly involved in the dispute, stipulates the strategies that must be obeyed by the players in a non-strictly competitive game, so that a special point in the negotiation set **N** can be reached. The characteristic of that point is that it might be considered impartial or just, and therefore produces a values of payoffs which are accepted in advance by the

⁴⁷ For example, one party could be an union an the other a Company; The negotiator could be representing its own interests or be a deputy.

⁴⁸ The arbitration schema delineated in this section refer to 2-person games. The solution of a n-person game involves the concepts of *imputation* (axioms of individual and group rationality), *characteristic functions* and *core* of a game, which are also associated to methods of coalition forming. Another method for solving n-person cooperative games is the **Shapley Value**, which in general gives more equitable solutions than the *core* does. The analysis of those topics are beyond the scope of this work, and a good introduction to the subject can be found in [WINS95].

⁴⁹ It should be emphasized that an arbitration scheme, in the present sense, regards a collection of rules that are expected to be followed by the players, aiming at the achievement of a solution for the conflict of interest, and it does **not** refer to exacting compliance from the contenders, which are supposed to abide to the norms prescribed by the umpire.

players. The *special point* is not always unique. But any alternative must obey certain conditions, in order to be considered equitable and free of ambiguities⁵⁰.

- i. The arbitrated solution of the game must be depicted by a point in \mathbf{N} , which means that each player's gain has to be at least the value that it would obtain when acting non-cooperatively;
- ii. The unit of measure used to build the players' utilities must have no influence whatsoever in the selection of the arbitrated solution;
- iii. The identity of the participants shall not act upon the contention, this property representing a symmetry⁵¹;
- iv. The variations in the solutions pointed out by an arbitration schema should be small when the differences among the games to which they refer are also small. In other words, the arbitrated solution points should not be subject to notable changes caused by negligible transformations \mathbf{A} of the payoff matrices⁵²;
- v. The potential capacity of the players to implement strategies that result extremely unfavorable to their opponents must be taken into account by the arbitration schema.

2.10.4 - Nash's Arbitration Scheme

In a bargaining problem, possible solution space may be represented by a convex closed region \mathbf{R} in the plane defined by the axes U and V , which denote the scales of the players' utilities. Every point (u, v) in that region corresponds to a pair of utilities associated to a specific agreement C . Given two feasible contracts C' and C'' , a lottery involving them is pictured by a line segment joining the points (u', v') and (u'', v'') . There exists in \mathbf{R} a point (u^0, v^0) that symbolizes the *status quo*, that is, the achievement of no compromise, denominated \mathbf{A}^0 .

⁵⁰ Mathematically, a generic and plausible arbitration scheme $f: \mathbf{N} \times \mathbf{s} \rightarrow \mathbf{N}$, where \mathbf{s} stands for the *status quo* preceding the realization of the game.

⁵¹ $f(T(\mathbf{N}), T(\mathbf{s})) = T(f(\mathbf{N}, \mathbf{s}))$, where $T(x, y) = (y, x)$.

⁵² $f(\mathbf{A}(\mathbf{N}), \mathbf{A}(\mathbf{s})) = \mathbf{A}(f(\mathbf{N}, \mathbf{s}))$.

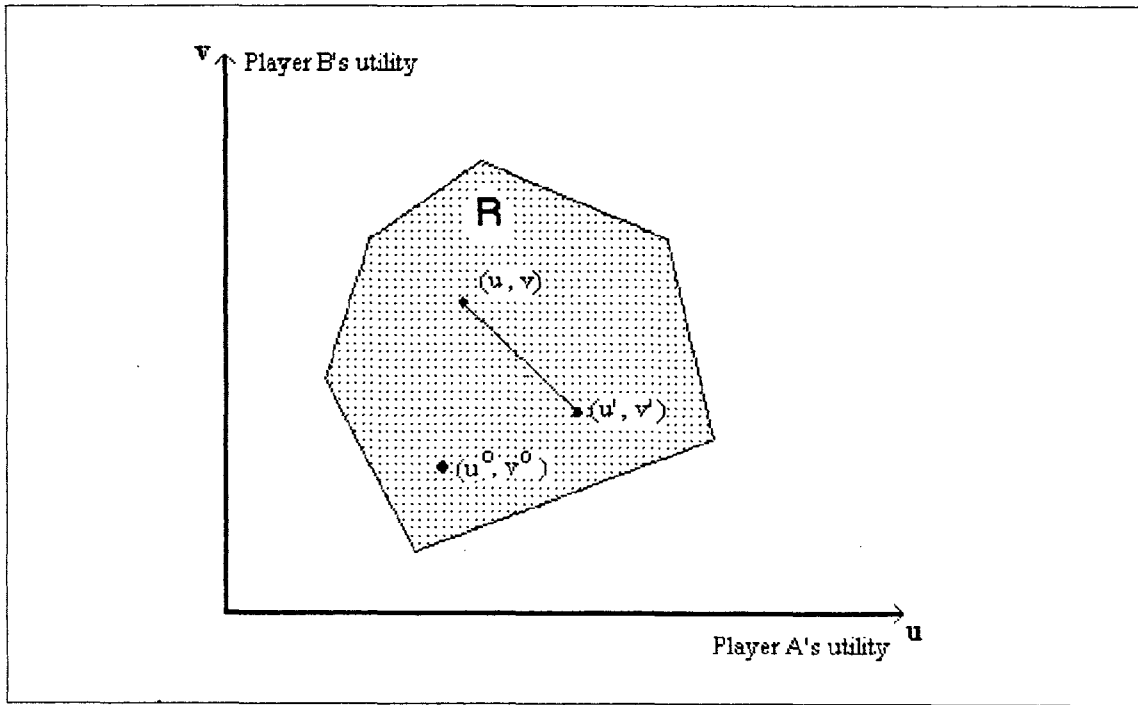


Figure 2.9 - Space of Feasible Agreements with the *status quo* point (u^0, v^0)

Nash's scheme objectives the determination of a function f that operates on $(R, (u^0, v^0))$ to reach another point (u^*, v^*) such that in a transformed region R' , obtained by the change of the axes' origin to (u^0, v^0) , the following relations verify:

$$(u^*, v^*) \in R', u^* > 0, v^* > 0; \quad \text{Eq. 2.25}$$

$$(u^* \times v^*) \geq (u \times v), \forall (u, v) \in R', u \geq 0, v \geq 0^{53}. \quad \text{Eq. 2.26}$$

The point $(u^* \times v^*)$ is called *Nash's Solution* to the bargaining problem $(R', (0, 0))$, which can be easily converted to $(R, (u^0, v^0))$. Nash's Solution is the only arbitration scheme that has the properties:

- i. Immunity to linear transformations of the utility scales;
- ii. It is optimal, in Pareto's sense (equal or greater payoffs than the *status quo*);
- iii. Independence of irrelevant alternatives⁵⁴;

⁵³ u^*v^* is the maximum of all products uv .

⁵⁴ Consider two games with regions R_1 and R_2 . If $R_1 \subseteq R_2$, and $f(R_2, (u^0, v^0)) \in R_1$, then $f(R_1, (u^0, v^0)) = f(R_2, (u^0, v^0))$. Put in words, if new contract possibilities are appended to a former bargaining problem in such way that it brings no change to the preexistent status quo, then, regarding the new setting, either the solution remains unaltered, or the new solution will belong to the added collection

- iv. Symmetry (Equal gains for symmetric players' roles).

2.10.5 - Criticisms to Nash's Scheme

Nash's model still is the *standard* approach to the treatment of conflict of interest between two players, especially when concerning economic problems ([ANDE89], [DOWR89], [HOEL90], [LESL90]). In spite of its prestige, Nash's scheme has been often criticized, predominantly because of (a) the *change of scales origin* feature ([MYER77], [KALA77]), (b) the observation of some divergence between theoretically obtained results and real life data ([NYDE74], [ROTH79]) and (c) the *irrelevant alternatives independence* axiom (ALEX92)].

Myerson [MYER77] also showed that in practical circumstances, evidence can be found that the players *do* accomplish interpersonal comparison of utilities, and this condition is excluded from Nash's solution by its compliance to the axiom of irrelevant independent alternatives.

2.10.6 - Other arbitration Models

Nash's axiomatic treatment has been widely employed by several other authors to modeling strategic negotiation processes, though with some alterations. Harsanyi, [HARS56], [HARS77], starting from a vintage Zeuthen's work [ZEUT30]⁵⁵, modified it in order that the bargaining procedure is accomplished through only two phases. In the last stage, the player which attains the littlest disadvantage in conceding the other's demand grants its request.

Rubinstein [RUBI82] devised another model where the negotiators propose a sequence of offers and counter offers, whose values are systematically reduced by a

of possible contracts. An arbitration scheme that abides to that prescription is defined as possessing independence to irrelevant alternatives.

⁵⁵ Zeuthen modeled the bargaining process as a sequence of small concessions from the player which incurs in the least significant loss when doing so. That method results mathematically equivalent to the maximization of the product uv in Nash's procedure, but with the advantage of attaching a plausible psychological component to the negotiation.

discount factor, what induces to obtaining a settlement. Although this process may be potentially infinite, Rubinstein proved that it has an unique perfect equilibrium⁵⁶, which is readily recognized by the players after the first bargaining phase.

Kalai and Smorodinski presented a strategic model of negotiation that implements a solution through a proportionality method. In that method, the axiom of irrelevant independent alternatives is replaced by another principle, called monotonicity axiom. It establishes that, if the negotiation set is enlarged in such way that when the gain of a player i increases, also does player's j , then the latter player's gain in the expanded game must not be inferior to the payoff it would achieve in the original configuration. The authors claim that this axiom, contrarily to Nash's, allows the participants to perform interpersonal comparison of utilities, what is more in accordance to the empirical evidences mentioned in [MYER77].

In a more recent work, Lootsma [LOOT89] proposed a methodology for the resolution of two-party conflicts using a procedure with two stages. In the first, each player examines its attitudes regarding possible concessions that could lead to a mutual agreement, and subsequently, the information obtained relative to the preferences and to the correlation between latent strategies is utilized to reach a reciprocal attractive contract. A criticism of Lootsma's approach was prepared by Bogetoff [BOGE92], where he argues that Lootsma only detailed the second phase of his model, and unduly assumed that the players always tend to express their true preferences in the first phase. This last supposition, although feasible when the contenders are looking forward to setting up a deal, is not realistic when the adversaries do not value sincerity or when the nature of the conflict overcomes that norm⁵⁷.

⁵⁶ A *perfect* (Bayesian) *equilibrium* is a strategy combination S and a set of beliefs μ such that at each node of the game: (1) The strategies for the remainder of the game are Nash (equilibria) given the beliefs and strategies of the other players; and (2) The beliefs at each information set are rational given the evidence appearing that far in the game. Using this idea, Kreps and Wilson introduced the concept of *sequential equilibrium*, applied only to games with discrete strategies, where a third condition imposes further restrictions on the players beliefs. From op. cit. [RASM89], p. 109-110.

⁵⁷ In bargaining processes, the submission of false indications regarding preferences is a sort of strategic behavior. In the majority of real cases, this conduct can be expected from the negotiators, except when the mediation plan accounts for incentives designed to curb such comportment. Bogetoff (op. cit. [BOGE92]) concludes that the cited *unfaithfulness* is likely to take place more frequently when the

2.11 - Cournot and Bertrand Games

2.11.1 - Cournot Theory

In 1838 Cournot proposed a new theory to explain the economic behavior of an oligopoly. The original problem considered the sale of water by two well owners in a small town. The product of each seller (water) is indistinguishable from one another, and hence both should be offered to the consumers for the same price. However, the price would fluctuate depending on the total quantity of water available daily. It consisted of the sum of the amounts q_1 and q_2 amount pumped out from the ground by the suppliers 1 and 2 in a given day [CASE79]. In this manner, the average price p should be a decreasing function f of $(q_1 + q_2)$, shown in Equation 2.27.

$$p = f(q_1 + q_2) = a - b(q_1 + q_2) \quad \text{Eq. 2.27}$$

Supplier i 's revenues r_i are

$$r_i = aq_i - bq_i(q_1 + q_2), \quad i = 1, 2. \quad \text{Eq. 2.28}$$

Making $q_i = \frac{a \cdot x}{b}$ and substituting in Equation 2.28, comes:

$$r_1 = \frac{a^2}{b} \times (x - x(x + y)) = \frac{a^2}{b} \times \pi_1(x, y) \quad \text{Eq. 2.29(a)}$$

$$r_2 = \frac{a^2}{b} \times (y - y(x + y)) = \frac{a^2}{b} \times \pi_2(x, y), \quad \text{Eq. 2.29(b)}$$

where x, y are the players' actions and the function π_1 and π_2 correspond to the way the payoffs of the game G^{58} are related to the players' actions x, y , which must be non-negative numbers.

G can be concisely represented by the players' following *objective functions*⁵⁹:

participants are pessimistic in relation to the achievement of a satisfactory arrangement prior to the outset of the process.

⁵⁸ G is a Cournot game played by the well owners 1 and 2.

- Player 1: *maximize* $\pi_1(x, y)$, s.t. $x \geq 0, y \geq 0$;
- Player 2: *maximize* $\pi_2(x, y)$, s.t. $x \geq 0, y \geq 0$;

Differentiating $\pi_1(x, y)$ and $\pi_2(x, y)$ using Equations 2.29 (a) and (b), comes

$$\frac{\partial}{\partial x} \pi_1(x, y) = \frac{\partial}{\partial x} (x - x(x + y)) = 1 - 2x - y \quad \text{Eq. 2.30(a)}$$

$$\frac{\partial}{\partial y} \pi_2(x, y) = \frac{\partial}{\partial y} (y - y(x + y)) = 1 - x - 2y, \quad \text{Eq. 2.30(b)}$$

which by making $\frac{\partial}{\partial x} \pi_1(x, y) = 0$ and $\frac{\partial}{\partial y} \pi_2(x, y) = 0$ yields $x^*(y) = \frac{1-y}{2}$ and

$y^*(x) = \frac{1-x}{2}$. The optimal action for player 1 is depicted by Equation 2.31.

$$\pi_1^*(x, y) = \frac{(1-y)^2}{4} \quad \text{Eq. 2.31}$$

Figure 2.10 illustrates the *reaction functions* $x^*(y)$ and $y^*(x)$. The term *reaction*⁶⁰ comes from the fact that the convergence path is delimited by those functions, tending to the point (x^*, y^*) . That point, located where the lines cross, is denominated the **Cournot-Nash equilibrium**. A property detained by the (x^*, y^*) equilibrium is *stability*, because a player's best response to another opponent's move always closes in upon it⁶¹.

⁵⁹ Strict maximization of the players' objective functions in G is impossible without a coalition or cooperation between the participants; Nevertheless, each player can try to make its revenues as large as possible.

⁶⁰ The name reaction must not be understood literally, since the participants of the game do not act in a sequential way, but simultaneously instead. Thus, a player has no opportunity to mull over the opponent's move in order implement its own. As a matter of fact, the reaction functions depict the manner how would the process evolve, in case the moves were successive.

⁶¹ The *stability* feature present in the Cournot-Nash equilibrium does not necessarily always occur in Nash equilibria found in other games.

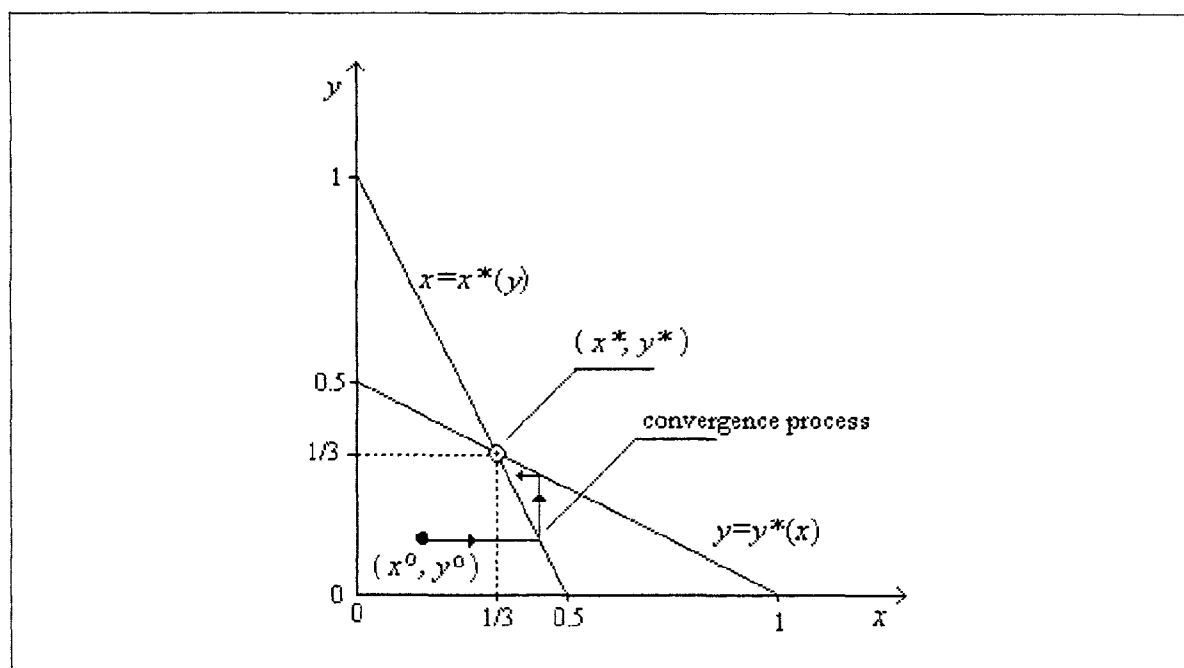


Figure 2.10 - Reaction Functions in the Cournot Duopoly Game

A variation⁶² of the Cournot game occurs when the rule regarding simultaneous (and therefore previously unknown) moves is relaxed, allowing that one of the players be able to know in advance which was the adversary's action. This conjuncture leads to another sort of counterpoise, called the **Stackelberg equilibrium** [STAC52].

2.11.2 - Bertrand's Model

The theory developed by Cournot remained virtually unnoticed for about half-century, and in 1883 it was reviewed by another French mathematician, J. Bertrand⁶³. An objection raised against Cournot's model was that the sellers might actually control both *quantities* (by accumulating inventories of the product) and *prices*. In this way, Bertrand proposed an alike model, whose solution (as currently known) was based on a Nash equilibrium in prices, rather than in quantities. In spite their similarity, the conclusions derived from each version resulted entirely different.

⁶² Cournot's model can be extended to n-players (op. cit. [CASE79]).

⁶³ Almost at the same time, Cournot's work was discovered by other economists (Walras, Edgeworth et al.), who divulged the theory to the economic community.

In Bertrand's game, the strategy space are the prices that each seller decides to charge for its merchandise, that are still indistinguishable. Once they fix their prices, the players will try to sell as much as possible, with no production capacity constraints⁶⁴. Furthermore, it is assumed that the buyers are completely aware of the prices offered by the competitors, and, in the duopoly situation, they will buy all the quantity they need only from the supplier which has the smaller price. The Equations that represent Bertrand's game follow from the foregoing Cournot's theory.

- ◊ If $(p_1 < p_2) \Rightarrow q_2 = 0, q_1 = D$ (total market's demand);
- ◊ If $(p_1 = p_2) \Rightarrow q_1 = q_2 = \frac{D}{2}$;
- ◊ If $(p_1 > p_2) \Rightarrow q_1 = 0, q_2 = D$.

Using Equations 2.27 and 2.28, the quantities sold by player 1 are:

$$q_1 = \frac{a - p_1}{b} = D, \text{ when } p_1 < p_2 \quad \text{Eq. 2.32(a)}$$

$$q_1 = q_2 = \frac{a - p_1}{2b} = \frac{D}{2}, \text{ when } p_1 = p_2 \quad \text{Eq. 2.32(b)}$$

$$q_1 = 0, \text{ when } p_1 > p_2 \quad \text{Eq. 2.32(c)}$$

Accordingly, player 1's revenues (payoffs) are given by Equations 2.33(a)-(c).

$$r_1 = p_1 \times \left(\frac{a - p_1}{b} \right), \text{ when } p_1 < p_2 \quad \text{Eq. 2.33(a)}$$

$$r_1 = r_2 = p_1 \times \left(\frac{a - p_1}{2b} \right), \text{ when } p_1 = p_2 \quad \text{Eq. 2.33(b)}$$

$$r_1 = 0, \text{ when } p_1 > p_2 \quad \text{Eq. 2.33(c)}$$

⁶⁴ This characteristic of Bertrand's original model is quite unrealistic, and output rationing can be assumed. See section 2.11.3.

In the Bertrand game, taking into account that a seller captures the total market demand if its price is only slightly inferior to the competitor's, this situation originates a unique Nash equilibrium of both prices equal to *zero*. Why? The answer is based on the previous assumptions⁶⁵ and on the very essence of Nash equilibrium, which is a position from where no player has any incentive to deviate, provided the other(s) remain fixed.

One strange characteristic of Bertrand's model is that a seller profit can abruptly change from a market share of 0% to 100% after even a small price cut, provided that it becomes lesser than the other player's. To downgrade this lesser-than-realistic feature, some modifications have been proposed for Bertrand's model, that have some impact in Cournot's method as well. They are briefly mentioned in the next section of this work.

2.11.3 - Some Variations on the Bertrand Game

1. Capacity Constraints

In Bertrand's approach, if the production capacity of the players is limited to L , it could be the case that the seller with the low price entices more customers than it can supply, because all will try to buy from it. Under that hypothesis, there will be a fraction of the market that shall be forced to deal with the high-price competitor⁶⁶. While the specific rationing rule employed to divide the consumers into two groups is unimportant regarding the low-price seller's profits, it has a great influence on the other's revenues.

Three of the most common rules are [RASM89]:

⁶⁵ No production costs, no product differentiation, consumers' total knowledge of the market prices and willingness to buy only from the seller that charges less.

⁶⁶ Provided that no consumer may abdicate of being served.

- *Intensity Rationing*: The buyers with more intense demand (values the product most) make the greater efforts to buy from the low-price seller. The competitor stays with the residual demand, that is, $(D-L)$.
- *Inverse Intensity Rationing*: The opposite of the former situation occurs. Those consumers which are mostly willing to purchase the item but not to wait to be served, buy the expensive product, and the rest of the demand ends up the low-price firm.
- *Proportional Rationing*: Every consumer has the same chance of buying from low-price supplier. That means that if M buyers want to pay the low price ($M > L$), then $\frac{M-L}{M}$ of each type of customers (M ; $(D-M)$) will be compelled to pay the high price.

One consequence of the capacity constraint is that the former Nash equilibrium for the price $(0, 0)$ does not hold any longer. Indeed, there is no pure strategy Nash equilibrium⁶⁷ in that variation, but a mixed strategy Nash equilibrium can be determined [LEVI72], [DASG86].

II. Product Differentiation

The reason why the classical Bertrand's model leads to a $(0, 0)$ price Nash equilibrium is the complete buyers' indifference to the quality of the products offered by either supplier. Thus, if one of the prices is only slightly inferior to the other, it is a sufficient motive for that seller to absorb the whole demand, if the production capacity is not constrained. However, if product differentiation is considered, or if the consumer's do not detain complete and perfect information on the prices being charged, the equilibrium becomes different. The quantities to be sold by each firm now will depend from both prices that are present in the market, p_1 and p_2 , and can be depicted by the linear functions of Equations 2.34(a) and (b).

⁶⁷ That circumstance is known as the *Edgeworth Paradox* [EDGE97].

$$q_1 = a' - b'p_1 + c'p_2 \quad \text{Eq. 2.34(a)}$$

$$q_2 = a' - b'p_2 + c'p_1 \quad \text{Eq. 2.34(b)}$$

The difference between the price coefficients b' and c' represent the degree of the product brands' substitutability. In those circumstances, the resulting payoffs are

$$r_1 = p_1(a' - b'p_1 + c'p_2) \quad \text{Eq. 2.35(a)}$$

$$r_2 = p_2(a' - b'p_2 + c'p_1). \quad \text{Eq. 2.35(b)}$$

Maximizing player 1's payoff by making $\frac{\partial r_1}{\partial p_1} = 0$, one arrives at its *reaction function*.

$$p_1 = \frac{a' + c'p_2}{2b'} \quad \text{Eq. 2.36}$$

Player 2 has a parallel condition, so the equilibrium price shall be $p_{\text{equil}} = p_1 = p_2$.

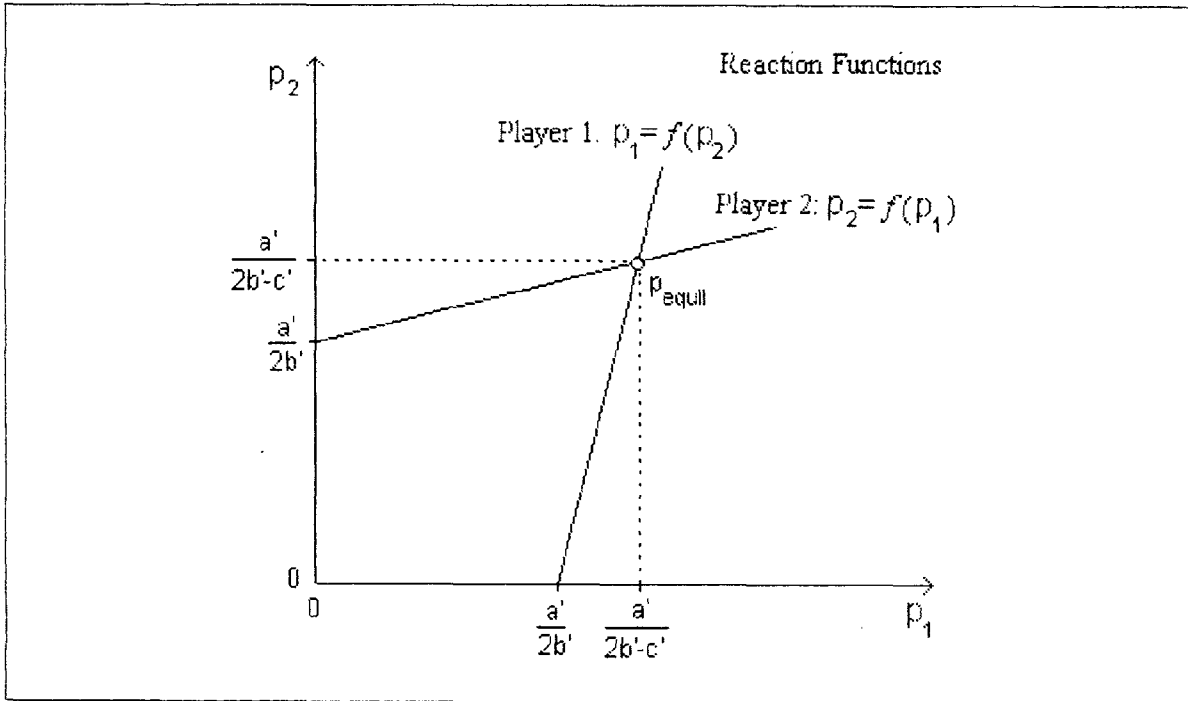


Figure 2.11 - Bertrand Reaction Functions and Equilibrium Price with Differentiated Products

2.11.4 - Concluding Remarks

The applications of game theory to economics are numerous and regard many diversified problems. For example, one could mention location models [GREE75], switching consumers [KLEM87], predatory pricing [KREP82] and others. The review of all those models involves extensive details and is beyond the scope of this dissertation.

Chapter 3

The Prisoner's Dilemma Game: A Survey

3.1 - Introduction

In 1950, scientists Melvin Dresher and Merrill Flood, from Rand Corporation, thought up a little odd game, that later became known by the name of "Prisoner's Dilemma". This game, according to some authors, has become the most famous of all nonzero-sum games [BACH77].

The denomination "Prisoner's Dilemma" (PD) was first given by Albert Tucker¹, another Rand Consultant. In May 1950, attending an invitation from the Department of Psychology of Stanford University to give a lecture on Game Theory, Tucker decided to discuss the game that Dresher had recently shown him. Considering the little background that the audience of psychologists had in Game Theory, he invented the anecdote of the prisoners. Flood, when confronted with the question of whether he realized the importance of the PD at the time it was introduced, he responded that "I never foresaw the tremendous impact that this idea would have on science and society..."[POUN92].

Many PD-like situations have been mentioned in the literature since ancient times². Nevertheless, none of the referenced archaic instances can be considered a valid precursor to the modern PD as defined by Dresher and Flood. Since its inception, a lot of research on the PD has been done, because it is a mathematical construct and mirrors many real

¹ Albert Tucker was a Princeton mathematician and knew both von Neumann, who pioneered in the formal and systematic mathematical approach of Game Theory, and John Nash, who was Tucker's former student and became one of the winners of the 1994 Nobel Prize in Economic Science.

² Early versions of the "Golden Rule"— "In everything, do to others what you would have them do to you", attributed to Jesus in the Gospel of Matthew, appear in the writings of Seneca (4 B.C.-65 A.D.), Aristotle (384-322 B.C.), Plato (427?-347 B.C.) and Confucius (551- 479 B.C.). Also in Hobbes' "Leviathan" (1651), the Prisoner's Dilemma turns up [POUN92].

world situations where a conflict of interest exists. Price wars between competing companies, the Cuban missile crisis in 1962, the arms race, and overpopulation problems are examples of PD-like circumstances [KAND96]. Also, biology[SMIT82], ecology [CAST95], evolutionary studies [SIGM93], sociology [GLAN94] and behavioral sciences [RAPO65], [NOWA95] have also benefited from the PD paradigm. Therefore, the methodology dealing with the PD has received a great deal of research attention. As a matter of fact, as early as 1971, close to 800 papers have already been published about the PD³ [RAPO74].

The story of the prisoners, as originally conceived by Tucker, involved monetary prizes and punishments rather than prison terms, and goes like this:

Two men, charged with a joint violation of law, were arrested and held by the police in two different cells, from where they cannot communicate with each other. They are told, one by one, that:

- 1) If one confesses and the other does not, the former will be given a reward, and the latter will be fined;*
- 2) If both confess, then both will be fined;*
- 3) If neither confesses, no evidence is gathered, so both go clear.*

Over the years, the Prisoner's Dilemma has been described with other "dressings", but its essence remains unaltered. One of the most absorbing versions was contributed by Douglas Hofstadter [HOFS85]. In that story, instead of prisoners, there are two dealers who agree to trade some merchandise for money, and each agent is expected to leave its part of the deal in different places, known to both. The dilemma is: Should one *cooperate* and abide to the agreement, whence making a reasonable profit from the transaction? Or should one *defect*, failing to fulfill its admitted responsibility in the deal and greatly increasing its gains by collecting the other's share without reciprocating?

³ In a posterior survey, the number of articles on the PD rose to approximately 2000 in 1977, and since the now famous experiments on this subject conducted by Axelrod [AXEL80] were divulged in 1980, there has been a surge in the quantity of scientific publications on the theme.

A single iteration of the traditional PD involves two players, and is characterized, in its normal form, by the general payoff matrix depicted in Figure 3.1.

| | | Player B | |
|----------|-----------------------------------|-----------------------------------|-------------------------|
| | | Cooperate (C) (Do not Confess) | Defect (D) (Confess) |
| Player A | Cooperate (C) (Do not Confess) | (R, R) | (S, T) |
| | Defect (D) (Confess) | (T, S) | (P, P) |

Figure 3.1 - Generic Payoff Matrix of the Prisoner's Dilemma

To constitute a Prisoner's Dilemma, the values represented by T, R, P and S in the cells of the matrix must obey the following relations:

$$\text{i. } T > R > P > S; \quad \text{Eq. 3.1(a)}$$

$$\text{ii. } R > \frac{T+S}{2} \quad \text{Eq. 3.1(b)}$$

$$\text{iii. } \frac{T+S}{2} > P. \quad \text{Eq. 3.1(c)}$$

The values often adopted for the payoffs are $T=5$, $R=3$, $P=1$ and $S=0$. If the game is to be played a single time, only the relation (i), ranking the payoffs in order of preference is needed to characterize the PD. However, if the game is going to be played repeatedly —thus constituting an *Iterated Prisoner's Dilemma - IPD*—, then the cardinal valuation of the gains must be designated. Condition (ii) is necessary to avert the prospect that the players' eventual strategy of mutually alternating between C and D may result more advantageous than playing reciprocal C's in a row⁴. Condition (iii) is less frequently

⁴ Alternating C's and D's advantageously in terms of the resulting sum of payoffs would transform the dispute in a *Coordination Game*. There, the players may use correlated strategies, (that might be reached

mentioned in the literature, but it aims at characterizing the move DD as the worst possible in terms of the sum of the payoffs.

In its classical form, the PD is seen as a difficult choice problem because it gives rise to a striking contradiction between *rationality* and *commonsense reasoning*. The attempts to solve it involve mainly alterations of the rules, allowing pre-contracting schemes, threats and side benefits, bounded rationality of the players and reputation effects in the iterated version [BICC93].

The PD is also a true paradox, because the MINIMAX strategies of both players intersect in the lower left cell, that is, mutual defection (DD), which corresponds to the unique Nash equilibrium in the game. Neither player has any grounds to regret the MINIMAX choice, because it is also the dominant strategy for both, therefore appearing that this option (DEFECT) is in the contenders' best interest, independently of which decision the opponent might make. What really happens, however, is that each player, by selecting this individually "rational" move (D), ends up with the payoff 1, hence worse than it would, should both depart from rationality and *cooperate* (C), in which case the gain for the two would be 3, according to the usual numeric values of the payoff matrix.

The gist of the paradox in the PD is just the strife between *individual* and *collective* rationality. And what is more impressive is that when the pair of players *together* forsake their greed and choose *cooperate*, the outcome will be better for both *individually*, not only in the aggregated sense of the collective.

The quantity of available scientific material related to the Prisoner's Dilemma is very large, ranging from purely theoretical developments [HUBE93], [GRIM94], to operations research [FOGE93a] and political science [HARD82]. The subsequent sections of this chapter provide a review of some of the most salient and up-to-date aspects of the research that has been developed on the Prisoner's Dilemma, which will include the following general topics:

by means of a randomizing device), which bring about a correlated equilibria. See Chapter 2 of this work and op. cit. [RASM89], p. 32-37.

- The Prisoner's Dilemma in the context of 2×2 games;
- Axelrod's computational tournaments;
- Evolutionary, Spatial and AI techniques approaches of the PD;
- The One-sided PD;
- Applications of PD-related games to economic problems;

3.2 - The Prisoner's Dilemma in the context of 2×2 games

The traditional PD, as formulated by Flood and Dresher and obeying the conditions stated by Equations 3.1(a)-(c) is only one of a large family of games. As a matter of fact, each possible type of 2×2 games is straight connected to the valuation of the payoffs T, R, P and S. If only the ordinal ranking of the payoffs is acknowledged, 78 distinct games can be formed, which are differentiated from each other by the players' strict preference ordering over the four outcomes⁵ T, R, P and S [RAPO66].

In order to provide a more convenient and formal representation of general 2×2 games, let us designate the actions COOPERATE and DEFECT as a_0 and a_1 , respectively, for the row player (A), and b_0 and b_1 , for the column player (B).

3.2.1 - Some Peculiar Social Dilemmas

If a 2×2 game is symmetric, which is equivalent to saying that both players agree in the order of preference of the payoffs, there are 24 possible rankings⁶ of the payoffs which are determined by the feasible pairs of actions a_0b_0 , a_0b_1 , a_1b_0 , a_1b_1 , not considering the existence of "ties". Not all of these games are dilemmas, since in many of them a simultaneously advantageous strategy can be easily found by both players. To pose a true dilemma, the following restrictions apply:

- $a_0b_0 \succ a_0b_1$;
- $a_1b_0 \succ a_1b_1$;

⁵ A "tie" is a situation where the players may manifest some type of indifference between two or more possible outcomes. If ties are allowed in the payoffs, like, for instance, the ordering $T \succ R \equiv P \succ S$, 726 games are possible [GUYE68].

⁶ This quantity corresponds to $4!$, the number of permutations of a_0b_0 , a_0b_1 , a_1b_0 , a_1b_1 .

- iii. $a_1b_0 \succ a_0b_0$;
- iv. $a_1b_1 \succ a_0b_1$

From the original list of 24 combinations, only four games remain, which have special denominations and are shown in Table 3.1.

| Ranking of preferences | Name of the Game |
|---|--------------------|
| $a_1b_0 \succ a_1b_1 \succ a_0b_0 \succ a_0b_1$ | Prisoner's Dilemma |
| $a_1b_0 \succ a_0b_0 \succ a_1b_1 \succ a_0b_1$ | Deadlock |
| $a_1b_0 \succ a_0b_0 \succ a_0b_1 \succ a_1b_1$ | Chicken |
| $a_0b_0 \succ a_1b_0 \succ a_1b_1 \succ a_0b_1$ | Stag Hunt |

Table 3.1 - Four Distinctive 2×2 Dilemmas

The four games of Table 3.1 represent quite prevalent real-life interactions among rational agents, and are much alike each other, although the Prisoner's Dilemma may be seen as a kind of "core", with the others occupying the outer layers [POUN92]. The Prisoner's Dilemma will be the object of an extended discussion along this chapter, so some comments are provided about the other related games.

- The *deadlock* game cannot be considered properly a true dilemma, and mutual defection (a_1b_1) is a Nash equilibrium. Both players do not really want to cooperate, each just wishes that the other does. The often observed failures in arms control treaties, for example, may be contemplated as a deadlock game.
- In *chicken*, the two players drive fast cars along a straight road, headed to each other in a route of collision. If both sustain the course (DEFECT), a mutual destruction happens for sure, which is obviously the least desired outcome for either player. Each one prefers that the other swerves (COOPERATE), which means assuming the *chicken* role (but still preferable to dying). Mutual cooperation (both swerve), though not bad if the two players adopt it concurrently, has a payoff that is inferior to the defector's gain in the DEFECT- COOPERATE conclusion. There are

two Nash equilibria in this type of game, namely a_1b_0 and a_0b_1 , which, curiously, mean that each player would rather do the *opposite* of what the other does⁷.

- The *stag hunt*⁸ refers to a situation where two hunters, searching for food in a forest, can coordinate (COOPERATE) and succeed in capturing a stag, which, even being equally shared, provides the best gain for both. But each can also forsake an agreed compromise in the conjoint chase and act isolated (DEFECT), thereby being able to catch a hare, which represents a quicker and safer of obtaining food, but in a much smaller amount. In case the second player, who cannot determine what the other is doing, insists in the stag hunt, which he cannot accomplish alone, he gets nothing. Here, mutual cooperation (a_0b_0) is clearly the Nash equilibrium. The temptation to defect arises when the rationality of the other player is put in doubt, bringing the belief that he will abandon the compromise. This dilemma may be more relevant and intense when the *stag hunt* is played by large groups⁹.

3.2.2 - The Effect of Preferences in 2×2 Games

The classical PD assumes that both players are rational, in the sense their preference functions are identical, and that they decide their moves always seeking the maximum gain. These gains are represented by the payoff values that appear in the payoff matrix, which stand for the utilities that the contenders may obtain in a given iteration. These numeric values of the utilities are fixed, and the players' preferences are ranked from the greatest to the smaller number.

An alternative and instructive approach, usually not taken into account in the analysis of the conflict of interest modeled by the PD among individuals belonging to a diversified population refers to the *choice principle* of the players. If the utility functions vary, it may well be the case that the dilemma dissolves. The dissolution is caused by the

⁷ A multi-person version of the *chicken* game is called the *volunteer's dilemma*, in the sense that some tough job must be done, and it will benefit all. But everybody would be equally in trouble if nobody enrolls.

variations in the players' predilections over the payoffs. The new resulting games, though maintaining the numeric values of payoffs, depart from the original PD.

Note that the changes in preferences do not occur in the absolute sense. Rather, they are related to the existence of an opponent and regard the *conjoint* outcome of the game. Frohock [FROH87] recalls that

"... Prisoner's Dilemma is a game played in conditions of uncertainty, and this uncertainty can extend to knowledge of what principles the opposing player is using. If compatible principles are found in the Prisoner's Dilemma, then a Pareto outcome is the saddle point¹⁰. But like principles may not be compatible, and different principles (compatible and opposed) are possible conditions of Prisoner's Dilemma games. Even in the original game the importance of pairings is clear: egoists play to a suboptimal outcome, altruists play to an optimal equity outcome. If structures are formed from compatible pairings, no conflict between individual and collective rationality occurs."

Frohock describes seven types of "personalities", that are listed with their ordering of preferences in Table 3.2.

⁸ The denomination "stag hunt" is owed to the philosopher Jean-Jacques Rousseau, who cited it as a metaphor in *A Discourse on Inequality* (1755).

⁹ If some benefit can be achieved only by the actuation of the majority involved (but excluding the non-participants), then the efforts consumed by the minority are wasted. If the non-participants may also enjoy the benefit, then a *free-rider* situation occurs [HARD82].

¹⁰ See Chapter 2 of this Dissertation - *A Review of Fundamental Topics of Game Theory*.

| Type of Player | Description | Ranking of Preferences |
|-----------------|--|---|
| Egoist (E) | The traditional PD player: seeks exclusively the satisfaction of its self interest | $a_1b_0 \succ a_1b_1 \succ a_0b_0 \succ a_0b_1$ |
| Benevolent (B) | Acts to maximize the opponent's gain. | $a_0b_1 \succ a_0b_0 \succ a_1b_1 \succ a_1b_0$ |
| Masochist (Ma) | Prefers the least possible return for itself. | $a_0b_1 \succ a_1b_1 \succ a_0b_0 \succ a_1b_0$ |
| Moralist (Mo) | Wants the highest and equal gains for both. If equality is not feasible, favors the adversary. | $a_0b_0 \succ a_1b_1 \succ a_0b_1 \succ a_1b_0$ |
| Utilitarian (U) | Fancies the highest total gain. | $a_1b_0 \succ a_1b_1 \approx a_0b_0 \succ a_0b_1$ |
| Sadist (Sa) | Chooses to minimize the other's gain. | $a_1b_0 \succ a_1b_1 \succ a_0b_0 \succ a_0b_1$ |
| Envious (En) | Gives priority to the greatest difference between its payoff and the opponent's. | $a_1b_0 \succ a_1b_1 \approx a_0b_0 \succ a_0b_1$ |

Table 3.2 - Types of Players and their Ranking of Preferences

Depending on the choice of the pairing of players, 28 distinct situations can betide. In several of them, no conflict of interest arises at all, because the priorities each player regarding the desired outcomes convey to mutual contentment.

The elective strategies for every possible combination of players' types are depicted in Table 3.3. The strategies marked with (★) consign those outcomes which result from a flawless rapport between the players' proneness.

| Matching | Joint Elective Strategy | Matching | Joint Elective Strategy |
|----------------|-------------------------|----------------|-------------------------|
| $E \times E$ | a_1b_1 | $B \times Mo$ | a_0b_0 |
| $B \times B$ | a_0b_0 | $B \times U$ | a_0b_0 |
| $Ma \times Ma$ | a_0b_0 | $B \times Sa$ | $a_0b_1 (\star)$ |
| $Mo \times Mo$ | $a_0b_0 (\star)$ | $B \times En$ | $a_0b_1 (\star)$ |
| $U \times U$ | $a_0b_0 (\star)$ | $Ma \times Mo$ | a_0b_0 |
| $Sa \times Sa$ | a_1b_1 | $Ma \times U$ | a_0b_0 |
| $En \times En$ | a_1b_1 | $Ma \times Sa$ | $a_0b_1 (\star)$ |
| $E \times B$ | $a_1b_0 (\star)$ | $Ma \times En$ | $a_0b_1 (\star)$ |
| $E \times Ma$ | $a_1b_0 (\star)$ | $Mo \times U$ | $a_0b_0 (\star)$ |
| $E \times Mo$ | a_1b_0 | $Mo \times Sa$ | a_0b_1 |
| $E \times U$ | a_1b_0 | $Mo \times En$ | a_0b_1 |
| $E \times Sa$ | a_1b_1 | $U \times Sa$ | a_0b_1 |
| $E \times En$ | a_1b_1 | $U \times En$ | a_0b_1 |
| $B \times Ma$ | a_0b_0 | $Sa \times En$ | a_1b_1 |

Table 3.3 - Possible Pairings of Players and the Resulting Joint Elective Strategies

It is important to note that perhaps the greatest difference between a static PD (which is defined by a game played either only once or repeatedly, but with no memory of former iterations) and an iterated PD is that, in the latter, the structural change brings in an increasing insight about the other player’s attitudes and pattern of behavior. This means that the prior uncertainty (about the opponent’s preferences) that influences the decisions is progressively lessened by the knowledge that is acquired through the repetition of the process [FROH87].

3.2.3 - Other Related 2×2 Games

Even though the Prisoner's Dilemma is considered the best-known 2×2 game, several others within this category are regularly cited and discussed in the Game Theory literature as important models of conflicts of interest. In those versions, the players' actions are not necessarily described as a COOPERATE versus DEFECT choice, though they are still binary. Moreover, their payoff matrix do not comply to the requirements established for the PD. For this reason, those games diverge from the main subject of this chapter and do not pertain here. Yet, aiming the expansion of the bibliographic review on this important category of games, mentions to some outstanding instances of these representations were made in Chapter 2 of this Dissertation, as a framework for the presentation of essential *equilibrium* concepts of Game Theory.

3.3 - Computational Tournaments of the Prisoner's Dilemma

3.3.1 - Overview

The scientific curiosity about the frame of decisions, choices and strategies in the circumstances denoted by the Prisoner's Dilemma in its iterated version (IPD) has been increasing since 1979. In that year, Robert Axelrod, a political scientist working in the University of Michigan, conducted a computer tournament¹¹, where 14 entries submitted by game theorists from diversified areas (economics, biology, political science, mathematics, psychology and sociology), and an additional random rule¹² were ran against each other in a round robin contest. All the strategies were composed solely by combinations of the elementary actions COOPERATE (C) and DEFECT (D). Subsequently, a second tournament was designed and implemented and this time 62 players competed, some of them the same as before. All the contestants received a report with the previous results attained and the respective analysis.

¹¹ This tournament's results were published in [AXEL80a] and [AXEL80b]. The implications of the results obtained gave origin in 1981 to another paper [AXEL81] with the title "The Evolution of Cooperation", to which was awarded the Newcomb-Cleveland Prize by the American Association for the Advancement of Science in 1981.

¹² The random strategy consisted of equal chances of choosing COOPERATE or DEFECT in each move.

A basic notion that should be well understood by all participants was the irrelevance of the number of individually favorable iterations that each player would get. It was stressed that what really mattered in that contest was the overall fitness of each strategy in the described context, and so the strategies should be selected with this fact in mind .

Axelrod's studies of the IPD, his tournaments and further extensions of the subject, including later contributions, have had a great impact in the development of game theory altogether with its applications in a variety of scientific fields. They have been the origin of the present research work and were the starting point of this doctoral dissertation.

3.3.2 - Aspects of the Tournaments' Dynamics

Axelrod's project began with the desire to investigate when or in which circumstances a decision maker in a situation of conflict of interests should COOPERATE¹³ aiming mutual benefit or DEFECT, trying to get an edge over the others.

In his book "The Evolution of Cooperation" [AXEL84] Axelrod describes the rules and results of the computer tournaments of the IPD and analyzes the ecological evolution of strategies by means of a simple criterion¹⁴ of reproduction and extinction. The proportion of individuals using each kind of strategy is a measure of its success.

In that class of simulations the possibility of occurring *mutations* affecting the original rules was not allowed. This is the same as saying that, given an initial collection of rules, their specifications remain fixed, without any alterations during the simulation, and only the relative participation in the total population changes along the iterative process. The payoff matrix employed by Axelrod is represented in Table 3.4.

¹³ It must be noted that COOPERATE, in the IPD's context, means altruism, and should not be confused with the term "cooperative" used in game theory to admit the possibility of negotiations among players before the moves in a game.

¹⁴ The score attained by each strategy in accordance to the payoff matrix in the first round divided by the total score of all players is the measure of "success" of a rule and corresponds to the expected number of copies (using the same strategy) in the next generation.

| | | Column Player | |
|-------------|---------------|---------------|------------|
| Line Player | | Cooperate(C) | Defect (D) |
| | Cooperate (C) | R=3 ; R=3 | S=0 ; T=5 |
| | Defect (D) | T=5 ; S=0 | P=1 ; P=1 |

Table 3.4 - Payoff Matrix of Axelrod's Computational Tournaments of the IPD

The pairs of numbers in the cells of the matrix are the Line and Column Players' gains, respectively. The letters R, S, T and P correspond to the payoffs regarding the individual roles assumed in a single iteration, namely "Reward" (mutual cooperation), "Sucker" (deceived), "Temptation" (Profiteer) and "Punishment" (mutual punishment).

The tournaments' rules were the following:

- i. The players may not make enforceable threats or commitments. Therefore, each must consider that *any* strategy is available to the other, and there is no way to be sure of what will be done by the opponent until the game takes place.
- ii. The only information that the players can have are the previous results of the iterations performed.
- iii. *Refusal* is not permitted, so no player may get away from an iteration or force the other to do so.
- iv. The payoffs relative to a particular pair of elementary actions are constant.
- v. The only communication permitted to the players is that achieved by the sequence of moves in the games.

The first tournament consisted of 24 000 iterations, with each of the 15 strategists running 200 times against each other and itself. The winner was the strategy "TIT-FOR-TAT" (TFT)¹⁵, with the average score of 504 points. This strategy starts cooperating, and thereafter simply repeats the opponent's last move. The 8 best classified rules were all "nice", that is, never defected first. A curious fact observed from the final outcomes was the influence that two particular strategies, which ended in the bottom half, had on the relative ordering of the top half. Axelrod called them *kingmakers*, because they didn't perform very well for themselves, but made it possible to others to gather a higher number

¹⁵ Submitted by Anatol Rapoport, professor of the Department of Psychology, University of Toronto.

of points. The most important *kingmaker* was the strategy called DOWNING¹⁶, whose decision to opt by C or D was taken according to a predictive model it made of each opponent's behavior, which estimated the probability of its cooperation after having received a D. The final purpose of DOWNING was to maximize the accumulated payoff in the long run. When the IPD simulation began, DOWNING's appraisal of this probability was $\frac{1}{2}$; This induced it to always defect in the first two moves and allowed it to receive two P's on account of some wrongly modeled strategies. On the other hand, not every *nice* strategy was *provocable*, as explained below, for TFT and other similar rules lost many points when they met DOWNING. In the end, DOWNING finished in the 10th position, with an average score of 391 points.

The complexity of the 14 entries submitted, in terms of the size of the program which represented each strategy, ranged from 4 to 63 lines of code. Surprisingly enough, the winner "TIT-FOR-TAT" was the simplest of all.

According to the analysis of the results of the tournament, three characteristics had a crucial role in determining the success of a strategy:

- i. Being *nice*, that is, never defect first;
- ii. Being *provocable* or intolerant, and retaliate promptly when receiving a defection;
- iii. Being *forgiving*, which means having the ability to accept an *apologize* and restore mutual cooperation if an opponent also demonstrated to wish the same thing.

In addition to these three features, TIT-FOR-TAT still has another special characteristic, *clarity*. TFT was simple enough to be quickly recognized and well understood, what avoided misinterpretations and misperceptions from its opponents and, moreover, insured stable mutual cooperation when this was also the other player's aim.

It should be observed that TFT is not the best strategy in the absolute sense. Its performance is strongly context-dependent, and in the first round-robin tournament, there

¹⁶ Proposed by Leslie Downing, (Psychology), who had previously published a paper on the Prisoner's Dilemma [DOWN75].

would be three other rules which could do better than TFT, should they have been included in that group¹⁷.

The second tournament was performed within a completely different environment, not only because the diversity of participants, but also because the results of the first contest had been made public by means of the report sent to all new contestants. This account was used by the applicants to subside the formulation of their second tournament's strategies. Furthermore, the length of the game was not any more fixed, as prevailing originally. Instead, the number of iterations became probabilistic, governed by a parameter $w = 0.99654$, that stood for the chance that there would be still another round (involving all players)¹⁸. Five repetitions of the simulations were performed, and the average number of iterations per simulation cycle was 151. Quite unexpectedly, "TIT-FOR-TAT", which had been re-submitted by the same person, was again the winner¹⁹.

Although a detailed analysis of each strategy's success or failure was made, those results referred to a particular environment where the simulations occurred. So, in order to verify the general performance of each rule, new tournaments have been simulated.

The representativeness of every strategy in the next rounds was now function of the previous relative success, that is, how well it performed before. The players that were present in a group from which all pairwise iterations were accomplished formed a *generation*. By this method, the number of individuals using the same rule in the next generation was proportional to the product of the previous number of representatives and their score, which, by its turn, was the frequency weighted average amount of points in the iterations with the other rules²⁰.

¹⁷ Those were: "TIT-FOR-TWO-TATS" (answer with a **D** only after receiving two **D**'s in a row from an adversary); "LOOK AHEAD" (analogous to search algorithms in graph trees, resembling some techniques used in chess playing programs); "DOWNING*" (a revised version of the original DOWNING, where the tactics of playing **D** in the two initial iterations was inverted to **C**).

¹⁸ The parameter w mirrors the importance given to future games' payoffs, and it can also be seen as a **discount factor**, because the strategies were fixed.

¹⁹ Although TFT could have been surpassed by another rules in the first tournament, these same rules didn't succeeded in doing so in the second, on account of the changes in the environment that took place due to the presence of a new breed of contestants.

²⁰ The simulation had 1000 generations, and TFT finished with the greater relative frequency, around 14%. It should be noted that a good performance in the initial generations does not mean that a final success is

The described process simulates the *ecological* performance of the rules. In this approach, the best adapted strategies tend to proliferate, and those less fitted will become extinct. The ecological simulation deals only with changes in the frequency distribution of types, since the specification of every strategy remains fixed. On the other hand, in a *adaptive* process, the rules can be modified, by means of the introduction of new proceedings or through *mutation* of already existent rules. The dynamics of Axelrod's mentioned tournaments' regard the former category of simulations, but he also investigated the latter in a subsequent work, which is explicated in section 3.4.4-II.

With the results of his tournaments as basis, Axelrod proved the following propositions and theorem [AXEL81], [AXEL84], which turn out to be a essential aspect of his research:

- *Proposition 1*

If the discount parameter w is sufficiently large, there is no independent strategy that can be better than the others.

- *Proposition 2*

If $w \geq \max [(T-R)/(R-S), (T-R)/(T-P)]$, and if all players adopt the TFT strategy, the Iterated Prisoner's Dilemma has a Nash equilibrium, and in this case TFT is called a *collectively stable strategy*, which means that it cannot be *invaded*²¹ by any other rule if every member of the population uses it.

- *Characterization Theorem*²²

The interpretation of this theorem is that the move C or D implemented by a rule X in any point n of the sequence of iterations is completely defined by the game's previous history, in order to X be collectively stable. While the inequality above is not satisfied, X may choose C or D freely without loosing its collective stability. When a rule interacts with itself (X|X), the circumstances where the largest quantity

expected. This is so because when a rule runs against every other in the beginning and gets many points, it must keep the same competence with the surviving strategies, as those less apt get excluded of the game.

²¹ The concept of *invasion* in a ecological IPD expresses the proliferation, by copy, of more successful strategies, regarding the accumulation of points. The "invasion" takes place when the ability of a native strategy to gather points is surpassed by another.

²² A strategy X is collectively stable if and only if X plays D in the n -th iteration every time the number of points accumulated ($\forall n$) by an opponent Y until that moment is such that $V_n(Y|X) > V(X|X) - w^{n-1} \times [T + wP/(1-w)]$. For a proof, see op. cit. [AXEL81].

of points is amassed are when both are. Therefore, *nice* rules detain the greatest potential of keep on playing C, even when they are threatened by *non-nice* invaders. The important advice embedded in this theorem is that anyone willing to adopt a collective stable strategy should only cooperate when its possible to absorb an exploitation and still maintain its mightiness.

- *Proposition 3*

A strategy X that has a non-zero probability of cooperating in the first iteration can be collectively stable only when $w \geq (T-R)/(T-P)$.

- *Proposition 4*

A *nice* strategy is collectively stable only if it is also *provocable*, that is, retaliates immediately after receiving a defection.

- *Proposition 5*

The rule "ALL D" is collectively stable.

This means that a population W made up exclusively of "ALL D's" cannot be invaded by any other strategy Z that is implemented by a newcomer. However, if the invaders using Z come in clusters and discriminate their opponents by means of establishing a certain proportion p of all iterations that are implemented with members of their own species, invasion becomes possible. Hence, a p -cluster invades a population W if $pV(Z|Z) + (1-p)V(W|W) > V(W|W)$ ²³.

- *Proposition 6*

The smallest value of p necessary for a specific strategy to be able to invade an "ALL D's" population in a cluster is accomplished by the strategies called *maximally discriminating*. These rules eventually play C for the first time even with those opponents which have never cooperated before. From this point on, the mutual cooperation is established (a sequence of (C,C) takes place) only with a strategy that is behaving in the same way. Therefore, the *maximally discriminating* strategies never play C again with an "ALL D".

²³ Axelrod observes that the pairings are not supposed to be random, because if this were so, a small group of Z's would have little chance of interact with each other in an environment with a great majority of W's. Considering the values from the standard payoff matrix (T=5, R=3, P=1 and S=0) and $w=0.9$, a group of Z's succeeds in invading the original population W if as few as 5% of all its iterations are within the group. If, however, the pairings are random, a strategy Z thrives whenever its overall proportion q is above 1/17, with the other parameters as before. See op. cit. [AXEL84], pages 211-213.

- *Proposition 7*

If a *nice* strategy cannot be invaded by a lone newcomer which acts differently, it will be also immune to clusters of the same kind of individuals.

- *Proposition 8*

If a strategy is collectively stable, it is also *territorially stable*.

This proposition refers to the *spatialized IPD*²⁴, which regards the dynamics of systems that are modeled by regions made up of cells, whose occupants interact with their "neighbors" and imitate them, on condition that their performance is better. The *territorial stability* is associated with the immunity that a location has against modifications of its original strategy provoked by imitation of an invader. The specifications of what consists a neighborhood and the criteria that make imitation possible can vary, and have a great influence on the way a territory is occupied. In these kind of systems, the "nice" character of a rule does not assure that it will be not invaded, specially if the parameter w (the *shadow of the future*) is small²⁵.

Axelrod re-examined the problem of *cooperation* in the context of the IPD in a later paper [AXEL88]. That article reviewed the previous work and presented short commentaries on several topics, suggested as lines of further investigations, including:

- ♦ **Iterations:** Multiple players - "n-person PD" - NPD;
- ♦ **Moves:** Possibility of inclusion of one more elementary act: *get away*, meaning *refusal* to play, that can be spontaneous or forced (*ostracism*);
- ♦ **Payoffs:** How the solutions can be affected by changes in the values of the matrix;
- ♦ **Noise:** Accidental faulty implementation of strategies (*trembling hand*) or misperception (*blurred minds*)²⁶;
- ♦ **Shadow of the future:** Its implications in the strategies adopted and their equilibria when the length of the IPD is finite or the discount factor w varies with time;
- ♦ **Population dynamics:** Differences between Nash equilibria and ESS's²⁷;

²⁴ An extended discussion of this approach is given in section 3.4.2.

²⁵ Axelrod illustrates the invasion of a territory solely composed of TFT's "inhabitants" by ALL D's ; the IPD parameters used in this example were $T=56$, $R=2$, $P=6$ and $S=0$, with w small, equal to $1/3$.

²⁶ In a homogeneous population of TFT's, the *noise* affects considerably the results. Regarding the factor *noise*, Nowak [NOWA92] mentions that, for an error rate less than 1%, the accumulated gains by each individual (fitness) is reduced by about 25%.

²⁷ *Evolutionary Stable Strategies* - See section 3.4.1.

- ♦ **Population structure:** How the “clustering effect” can influence the possibility of a homogeneous population to be invaded by some other strategy.

Axelrod’s computational tournaments of the IPD and the insights that they brought can be considered indeed a landmark in the study and comprehension of the mechanisms that govern conflicts of interest and the benefits of mutual cooperation. In its foreword to the 1990 edition of Axelrod’s book *The Evolution of Cooperation*, Richard Dawkins²⁸ states that “*The world’s leaders should all be locked up with this book and not released until they have read it. This would be a pleasure to them and might save the rest of us.*”

3.4 - Evolutionary Games, Spatial and AI Techniques Approaches of the PD

3.4.1 - An Evolutionary Biological Application: The "Hawk-Dove" game

The first explicit application of Game Theory to evolutionary biology was given by R. Lewontin [LEWO61]. His work, followed by numerous other related papers, examined how a species played against nature to minimize its probability of extinction. However, a much more interesting and important contribution on that area was provided by the biologist John Maynard Smith [SMIT82], who explored the way members of animal species play games against each other. In this case, both the population dynamics and equilibria are considered and analyzed, mainly by means of a special kind of game devised by Maynard Smith: The Hawk-Dove (HD) Game. Though not retaining the same structure of the PD’s payoff matrix, HD has various similar features regarding the choice of strategies, and was used to introduce the important concept of the *Evolutionary Stable Strategy* - ESS, that consists in a new class of equilibrium in games. Also, in Maynard Smith’s model, the criteria of *rationality* and *preferences* stated in classic Game Theory are replaced by those of *dynamic stability* of populations and *Darwinian adaptation* (“The survival of the fittest”).

In the Hawk-Dove game the players are considered to be individuals from animal species which, like the participants of a PD, have basically two elementary actions

²⁸ Author of *The Selfish Gene* (1976) and *The Blind Watchmaker* (1986), among other works.

available, namely Hawk (H) = Escalate or Dove (D) = Display, to be adopted one at a time. The dynamic evolution of the strategies employed is analyzed in the course of several generations, from the assumption that each species aims at maximizing its expected accumulated *adaptive gain*, which depends on the behavior of its adversaries. This adaptive gain refers basically to the possibility of free (without competition) exploitation of the resources offered by the environment where the species already lives or intends to establish itself.

The HD payoff matrix is represented in the Table 3.5.

| | | Column Player | |
|-------------|--------------|---------------|-------------|
| Line Player | | Escalate (H) | Display (D) |
| | Escalate (H) | 1/2 (V - C) | V |
| | Display (D) | 0 | V/2 |

Table 3.5 - The Hawk-Dove Game's Payoff Matrix

The values in the cells correspond to Line Player's adaptive gain. V is the payoff of *Hawk* (H), when the other plays *Dove* (D) and C consists of a loss, associated to a 50% probability that a H strategist suffers injuries when meeting another H. When a dispute takes place, the net expected adaptive gain is equally shared.

Assuming that when both players start a game they have an equal adaptive gain W_0 , the accumulated payoff by each species after the game (fitness) will be noted by $W(D)$ and $W(H)$, provided that the relative frequency of H's in the whole population of the species involved is p .

$$W(H) = W_0 + p E(H,H) + (1-p) E(H,D)$$

Eq. 3.2(a)

$$W(D) = W_0 + p E(D,H) + (1-p) E(D,D)$$

Eq. 3.2(b)

$E(.)$ is the Line Player's payoff in an iteration between any two type of strategists. Along the iterations, the players reproduce (create "offsprings"), which quantities are proportional to their respective achieved fitness.

Calling p' the proportion of H's in the next generation, the population dynamics is described by Equation 3.3.

$$p' = \frac{p \cdot W(H)}{\bar{W}}, \text{ where } \bar{W} = p \cdot W(H) + (1-p) \cdot W(D) \quad \text{Eq. 3.3}$$

Knowing the values of V , C and p_0 (initial proportion of H's), it is possible to determine the evolution of each species' relative frequency p and $(1-p)$. An inquiry that remains is to discover if *stable states* can exist, and if the answer is affirmative, which specific characteristics they have.

Definition: A strategy I is said to be *stable* if, being adopted by the majority of the population, its adaptive gain is greater than any other strategy, and so its invasion by *mutants* will not be possible.

Supposing that $p \ll (1-p)$ is a small proportion of mutants using strategy J , which interact with strategists I , the respective fitness are depicted by Equations 3.4(a) and (b).

$$W(I) = W_0 + p \cdot E(I,I) + (1-p) \cdot E(I,J) \quad \text{Eq. 3.4(a)}$$

$$W(J) = W_0 + p \cdot E(J,I) + (1-p) \cdot E(J,J) \quad \text{Eq. 3.4(b)}$$

If I is stable, $W(I) > W(J)$, and, as $p \ll 1$, it is necessary that at least one the conditions (a) or (b) below are verified, $\forall J \neq I$.

$$\text{a) } E(I,I) > E(J,I)$$

$$\text{b) } E(I,I) = E(J,J) \text{ and } E(I,J) > E(J,J)$$

Any strategy I which obeys the inequalities (a) or (b) is called **Evolutionary Stable (ESS)**²⁹, and it is such that if all members of a population employ it, no other rule will succeed in invading it and thrive [MOLA92]. In the "Hawk-Dove" game, D (Dove) is not an ESS, because a population of D 's can be invaded by H 's ($E(D,D) < E(H,D)$). On the other hand, H is an ESS if $1/2(V-C) > 0$ or $V > C$. In this case, for (H,H) the probabilities of

²⁹ The pre-conditions imposed by Maynard Smith for the existence of ESS's are: a) infinite population; b) the iterations are between two individuals at a time and these are indistinguishable of each other regarding strategies that will be selected before an iteration happens (symmetric games); c) the reproduction is asexual.

getting V or C are 50% for both players, and if $V > C$, it is worth running the risk of receiving an injury (C) to achieve the gain V ³⁰.

Besides mixed strategies involving H and D , several conditional behaviors are also considered in sequential HD games, such as:

- Retaliate: Start with D , but change to H if opponent escalates (adopts H);
- "Bully": Start with H and retreat to D if opponent escalates;
- "Bourgeois": Play H if owner (of the territory) and D if intruder;

Asymmetric games are also analyzed, thus relaxing one of the previous conditions assumed for the existence of ESS's. In this mode, the players are separated in categories according to some attribute which indicates strength, such as *bigger*, *better*, etc., while the other has opposing characteristics. Also, every player is perfectly aware of its own characteristics. However, the strategies that are employed are supposed not to be correlated either with this information or with the results of the previous iterations.

Reinhard Selten [SELT80] proved that if a game has at least one asymmetry which is *common knowledge* to the contestants, then it cannot have a *mixed* ESS³¹. Assuming that one or more asymmetries exists, another type of behavior is considered, called *Assessor*. An *assessor* (A) chooses H if it finds itself in a better position than its adversary, and selects D conversely. The acquisition of the information³² concerning the other's characteristics can be supposed to have a cost c ³³ for the *assessor* strategists. In a iterated game involving the strategies H , D and A , A is an ESS if (a) $c < \frac{1}{2}V$ and (b) $Cx > V(1-x)$; H is an ESS if (a) $c < \frac{1}{2}(V-C)$ and (b) $Cx < V(1-x)$, where x is the probability that the privileged player wins a (H, H) dispute. Alternate behaviors H and A cannot take place in an iterated game with the same values of the parameters V , C , c and x ,

³⁰ In the human case, this decision rule generally does not apply. Kahneman e Tversky, in their "Prospect Theory" [KAHN79] show empirical evidences that players' concerns for losses are greater than those for gains.

³¹ Taken in the traditional sense, as, for example, select an action A with probability p or another action B with probability $(1-p)$. In the case mentioned, however, *pure* ESS's can exist.

³² Presumed with no ambiguities. If there is a chance of errors in the *assessor's* process, Selten's theorem [SELT80] is no longer valid.

³³ $c < C$, that is, the cost c of acquiring the information about the adversary's strength is smaller than the loss C related to an eventual injury.

for if an *assessor* discovers that it is in inferiority³⁴, it gives up this strategy and chooses H or D depending on the mentioned parameters.

Maynard Smith notes that A has advantages that help it to be an ESS, because if c is small relative to C , the eventual privilege of a better relative position is a reliable indicator of success ($x \cong 1$) if a (H,H) dispute takes place. It should be observed that x does not need to approach 1 in order to A be an ESS. Even if $x < 0.5$, A is an ESS when $C \gg V$. This fact gives rise to a kind of paradox, because it means that a player uses H if inferior and D if in advantage.

Several other contingencies can have a role in the evolution of asymmetric games. One factor worth citing is the *Resource Holding Power* (RHP), which should be interpreted as the demonstration, by a player, of the amount of resources available to it that can influence the outcome of a dispute. The RHP is a kind of a *measure of power* and has a relevant effect on the behavior of an opponent that uses the *assessor* rule. Specifically, in the same way a cost c was assigned to information gathering, it can also be assumed that a similar expense will be associated to *power exhibition*³⁵ with the purpose of intimidating an opponent. How accurately (level of perception) a threat is represented by a displayed RHP is a factor equally important in the strategies' formulation and in their evolution³⁶.

Although the Hawk-Dove game had its original aim as a paradigm for the treatment of biologic problems, it has been quite influential in the general development of Game Theory. In fact, the work of Maynard Smith [SMIT82] marks the advent of *Evolutionary Game Theory* [WARN95]. Its significance in the area can be measured by the great number of citations in the specialized literature, e.g. [COLL91], [GODF92], [NOWA92], [NOWA93], [BEND93], [FOGE93], [SIGM93], [ATMA94], [FOGE95].

³⁴ Regarding the strength attribute, not the rule H or D.

³⁵ The "Power display" must not be backed by the existence of the advantages; it may be a "bluff".

³⁶ In human relations, a typical example of how misperception of RHP's can be hazardous to all refers to the international political situation known as "cold war". Each side involved in this kind of dispute attempted to show the maximum RHP possible (at high costs), so that no doubts might exist about its military retaliation power (MAD: *Mutual Assured Destruction* policy). This was so because an underestimate from the other side could take it into believing that "playing tough" would be advantageous, and this would almost certainly bring an escalation obviously feared by anyone.

3.4.2 - Spatial Models of the IPD

An spatialized model of the iterated Prisoner's dilemma has all the characteristics of the classical version, plus a new feature, that is the embodiment of the dimension *space* in the game [NOWA92(a,b)], [MAR93], [HUBE93], [NOWA94]. According to some rule, that can be random or not, the players are scattered in a *field*, usually a two-dimensional grid. There can be more than one manner by which the participants may interact. In the most common mode, the contestants compete with their "neighbors"³⁷, that is, those players which occupy adjacent cells in the grid in a round-robin tournament, where each plays once with one another. A completed round is called a *generation*, and at this point each cell copies the most successful strategy of its neighborhood. The evolution of such a spatial grid is dependent on the values of payoff matrix. Even if the binary IPD is played, with only pure *cooperators* and *defectors*, C and D actions allowed, quite complex dynamics may result. It is possible that cooperators are extinguished, but often shifting mosaics occur, with mixtures of pure cooperators and defectors coexisting indefinitely in unsteady proportions. Nevertheless, recent studies performed mainly by Nowak found that the average composition of the population is predictable [NOWA95].

Besides the method described above, other approaches using the basic spatial paradigm of the IPD appear in the literature. It was employed, for instance, in the philosophical issue of demonstrating the pervasiveness of the Gödelian notion of *mathematical undecidability* in more practical, less abstracts problems [GRIM94]³⁸. Another example is provided by Weeks et al. [WEEK96], who simulate a spatial IPD in a playing field with "walls" and random matchings. In this work, variable payoffs are considered and the players are represented by cellular automata machines, which evolve their strategies with the aid of a genetic algorithm.

³⁷ Those players which occupy adjacent cells, usually eight in a chessboard-like grid, or four, if diagonals are not included. An hexagonal-patterned field may also be considered.

³⁸ Grim [GRIM94] argues that some formerly pure mathematical philosophical concepts, like Gödel's *undecidability*, Rice's theorem in recursion theory, and recent Chaitin's theorem in information theory, which are coming closer to more practical applications in physics, engineering, economics and theoretical biology.

The spatial models of the IPD are proving extremely useful as a new method of simulating and analyzing how cooperative behavior and mutual help can influence species evolution and the coexistence of diversity when territorial location is significant. It has been aiding studies in environments that range from the molecular and genetic level to phenotypic and social structures [CAMP85], [NOWA95], [LLOY95].

3.4.3 - Singular Reactive Strategies

The strategies that decide their moves taking into account the opponent's previous actions (C or D) are denominated *reactive*. In their simplest form, those rules only consider the last iteration. The best known of those kind of strategies is the famous Tit-for-Tat (TFT), which proved quite effective and robust in Axelrod's computational tournaments of the IPD, and since then, also in many other subsequent inquiries on the subject. For this reason, TFT is regularly included in experiments regarding simulations of the IPD³⁹. It has been serving as a kind of a reference parameter in the investigation of other strategies' proficiencies. TFT has been used also as a basis for some related variations, as for example, the *Suspicious* TFT, whose only difference is that it starts defecting instead of cooperating.

Employing another type of reactive strategies, Nowak and Sigmund [NOWA92] analyzed the infinite IPD⁴⁰ under the aspect of ecological development performed by heterogeneous populations. In that model the players employ reactive strategies represented by $E_i = (y, p_i, q_i)$, where y is the probability that a player cooperates in the first move and p_i e q_i are the conditional probabilities of cooperating (C) after having received a C or D in the previous iteration, respectively.

Several simulations of this model of the IPD were accomplished, where 99 distinct reactive strategies (whose parameters p_i and q_i have been randomly chosen from uniformly distributed values in the interval $[0, 1]$) participated together with TFT* — *almost* TFT—

³⁹ In the *Fuzzy Iterated Prisoner's Dilemma*, whose model is presented and detailed in Chapter 5 of this Dissertation, the traditional (dichotomic) TFT has been included in the population of strategies, along with a gradual version (Fuzzy-TFT) and also another celebrated reactive strategy, PAVLOV.

⁴⁰ The probability of occurring a next iteration, in that case, is taken to be equal to 1.

(parameters $p_i = 0.99$ and $q_i = 0.01$ and $E_{TFT^*} = (0.99, 0.01)$)⁴¹. The payoff matrix was the same of Axelrod's tournaments. The determination of the relative frequency x_i' obtained by particular strategy was given by

$$x_i' = \frac{x_i f_i(x)}{f(\text{avg})}, \quad \text{Eq. 3.5}$$

where x_i is the frequency of i in the present generation, $f_i(x)$ the average gain of E_i in the existing population and $f_{\text{avg}} = \sum x_i f_i(x)$ is the average gain of the whole population.

The ecological evolution observed during 1000 generations was rather interesting. Starting the process with a diversified composition of strategies, around the 100th generation the rules that hardly ever played C ($E_{\text{ALLD}} \cong (0,0)$) added up to practically 100% of the population. By the 150th generation, the so-called *Generous* TFT⁴² — GTFT reappear. In the end, the winner was $E_{\text{GTFT}} = (0.99, 0.33)$. The authors comment that the final success of GTFT was only possible because a small group of TFT* survived near the 100th generation, when E_{ALLD} had almost taken over. Hence, in this case TFT* performed as a kind of *catalyzer*, making the victory of GTFT possible. In the generations above the 300th, the average payoff of an iteration was about 3 (the reward R for mutual cooperation), because at this time almost the entire population had already turned to E_{GTFT} 's, implying in a high probability (0.99) of moves (C,C).

An equally remarkable contribution on the field was also given by Nowak and Sigmund, regarding the robustness of TFT in *noisy*⁴³ environments [NOWA92], [NOWA93b]. Those researchers found that TFT does not endure in simulations which, mirroring real-life conditions, some mistakes are made [BEAR93]. Instead, another strategy, employing the criteria *Win-Stat*, *Lose-Shift*, succeeded in dominating the modeled

⁴¹ Because an infinite duration ($w=1$) is assumed, the memory of the first move will vanish, and so the parameter y can stay out of the notation of E_i .

⁴² *Generous* is the generic denomination of the strategies whose p_i and q_i values are near 1 and 0, respectively.

⁴³ The term *noise* refers to erroneous implementation or perception of strategies, e.g. *trembling hand* and *blurred minds*. See section 3.3.2.

populations in the long run. This reactive rule, dubbed PAVLOV, has a *modus operandi* as straightforward as TFT, and it can be translated in the following: If in the previous iteration both players made the same move (either CC or DD), PAVLOV cooperates. Otherwise (CD or DC), defect.

For its performance, PAVLOV counted on four main advantageous attributes:

- i. It is able to maintain cooperation (CC) and continuously receive the *reward* payoff;
- ii. It is *provocable*, defecting after receiving the *sucker's* payoff (CD);
- iii. It tries to restore mutual cooperation, playing C after a DD, but it is non-exploitable, because it turns to defection after an eventual failure in its objective.
- iv. It takes advantage of the exploitable partners, stockpiling points from all-out cooperators (DC).

PAVLOV has the ability of quickly correcting errors. One drawback is that it fails against an All-D⁴⁴, because in this case an alternation of DC's and DD's would occur.

Still more recently, in simulations of the IPD with variable degrees of noise, Wu and Axelrod [WU95] showed that the reactive strategy *Contrite* TFT⁴⁵ (CTFT) performed better than both PAVLOV and GTFT when the noise level exceeded about 1%. Those authors also argue against the robustness of PAVLOV in the investigated environments.

3.4.4 - The IPD with Finite Automata and AI-aided players

Lately, some lines of research regarding the performance and evolution of adaptive strategies in the IPD have been increasingly making use of automata and several AI related tools. The efforts in that direction include the use of Finite State Machines, Genetic Algorithms, Evolutionary Computing and Neural Networks. This section reviews a representative sample of these techniques applied to IPD simulations.

1 - The IPD played by a pair of Finite State Machines (FSM's)

⁴⁴ Nevertheless, a population composed of PAVLOV strategists cannot be invaded by All-D's if PAVLOV is slightly less cooperative after a P than after a R. This behavior is mirrored by the conditional probabilities of cooperating after each of the four possible previous outcomes, like, for instance $p(C|CC)=0.999$, $p(C|CD)=0.001$, $p(C|DC)=0.001$, $p(C|DD)=0.995$ [MIL93].

⁴⁵ CTFT cooperates after the other player had defected in response to a previous CTFT defection.

One of the first attempts to embed automatic decisions in IPD players mentioned in the literature was contributed by Rubinstein [RUBI86], who simulated an IPD supergame⁴⁶ with two FSM's. In each iteration the players selected moves that depended on the previous sequence of outcomes. A player's global strategy contemplated all possible histories of results, implementing decisions that have been formulated exclusively by its respective FSM.

An FSM can represent a higher-order Markovian Process, and the finite machines that took part in the cited supergame had the following characteristics:

- i. A finite number of states, with an initial one;
- ii. An output function, which defines the move (C or D) to be accomplished in the k^{th} iteration;
- iii. A transition function that determines the $(k+1)^{\text{th}}$ state depending on the k^{th} state and opponent's move.

In this model, the players' problem is reduced to the choice of their FSM's, since the moves are completely determined by the selected machines. Under those conditions it is also possible to find a pair of gains that is a Nash equilibrium⁴⁷.

Besides the attainable results, another concern is the degree of the complexity⁴⁸ of the chosen FSM's. A simple measure of this attribute is proposed, calculated only as a function which associates *operational costs* proportionally to the number of internal states of each FSM. The costs are independent from the utilization of the states during the game.

The type of FSM employed is a Moore Machine⁴⁹, denoted M_i for player i and represented by a four-tuple $\{Q_i, q_i^0, \lambda_i, \mu_i\}$, where

⁴⁶ A "supergame" is a game that is repeated an infinite number of times. Each iteration is also denominated a *period*.

⁴⁷ The PD does not have a *jointly admissible* pair of strategies; Therefore, the game is not soluble in the strict sense, though it is soluble in Nash's sense. This result extends to the iterated PD. See Chapter 2 of this Dissertation — *A Review of Fundamental Topics of Game Theory*.

⁴⁸ It is possible to specify more elaborated FSM's complexity measures, as those dependent on the automaton's type of the transition function.

⁴⁹ *Moore Machines* are automata whose output, at any time, is a direct function of the state to which the machine transited. A more general class of automata are the *Mealy Machines*, in which the outputs derive from the transitions that occur from state to state. The path followed (the sequence of states) until a particular state may be different, therefor influencing the output. A Mealy machine can be always logically replaced by a Moore automata, though requiring a greater number of states.

- Q_i is the finite set of internal states of M_i ;
- $q_i^0 \in Q_i$ is the initial state;
- $\lambda_i: Q_i \rightarrow S_i$, with S_i being the set of available strategies for player i ;
- $\lambda_i(q_i)$ is player i 's strategy whenever M_i is in state q_i ;
- $\mu_i: Q_i \times S_j \rightarrow Q_i$ ($i \neq j$), meaning that μ_i represents a transition function which defines the next internal state with basis on the previous state and opponent's move.

The choice of two particular FSM's M_1 and M_2 completely define the sequence of outcomes, as long as the introductory move is settled by a zero-length string (no results yet) and the initial internal state q_i^0 of the machines. For example, player i 's first move may be $s_i^1 = \lambda_i(q_i^1) \in S_i$. After j plays, M_i changes its internal state to $q_i^2 = \mu_i(q_i^1, s_j^1)$.

A diagram is a convenient way of depicting a Finite State Machine. Figure 3. portrays a FSM that uses the TFT strategy, with the characteristics $Q = \{q_C, q_D\}$; $q^0 = q_C$; $\lambda(q_C) = s$ and $\mu(q_C) = q_s$, for $s(\text{state}) = \{C, D\}$.

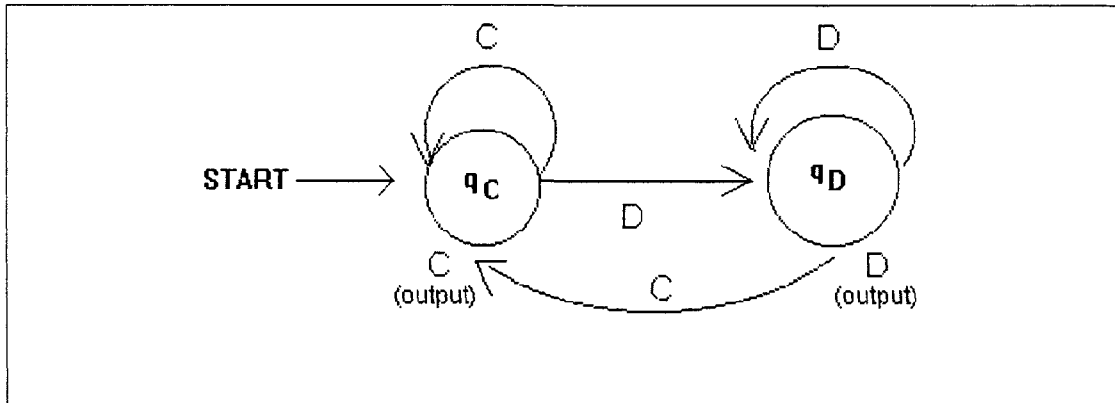


Figure 3.2 - A FSM that plays TFT in the IPD

After a certain number of iterations, being finite, the FSM's will necessarily repeat a pair of previously assumed states. Calling $q^t = (q_1^t, q_2^t)$, $t = 1, 2, \dots$ the sequence of states of M_1 and M_2 , there will come an occasion t_2 , $t_2 > t_1$ such that $q^{t_1} = q^{t_2+1}$. The period t_2 is the last of the *cycle*, because in t_2+1 the machines go back to the state they had in t_1 . The length or duration of the cycle is $T = t_2 - t_1 + 1$.

Two types of equilibrium are considered: Nash's and the *semi-perfect*, the latter concept introduced by Rubinstein in the paper. Both equilibria regard the players' ranking of preferences in the choice of the FSM's.

Defining the sequence of moves as $s^t=(s_1^t, s_2^t)$, $t=1, 2, \dots$, the average payoff π_i of player i per cycle is given by Equation 3.6.

$$\pi_i(M_1, M_2) = \frac{1}{T} \sum_{t=t_1}^{t_2} \mu_i(s^t) \quad \text{Eq. 3.6}$$

For a generic FSM, $M = \langle Q, q^0, \lambda, \mu \rangle$, $|M|$ denotes the number of states that exist in the set Q , and $M(q)$ is the machine whose initial state is q . Calling $>_L$ the lexicographic order in \mathbb{R}^2 , the definition in Equation 3.7 means that a player's ranking of preferences regarding a feasible pair of machines is established first, by the greatest average payoff it can achieve, and second, by the lesser number of states of its machine, provided the payoff is not inferior.

$$(M_1, M_2) \succ_i (M_1', M_2') \text{ if } \{ \pi_i(M_1, M_2), -|M_1| \} >_L \{ \pi_i(M_1', M_2'), -|M_1'| \} \quad \text{Eq. 3.7}$$

Two FSM's M_1^* and M_2^* are a Nash equilibrium if there is no other machine M_1 or M_2 that verifies either one of the relations depicted by Equations 3.8(a) and (b).

$$(M_1, M_2^*) >_1 (M_1^*, M_2^*) \quad \text{Eq. 3.8(a)}$$

$$(M_1^*, M_2) >_2 (M_1^*, M_2^*) \quad \text{Eq. 3.8(b)}$$

The main definition of the paper regards the *semi-perfect equilibrium* (SPE), which asserts that a SPE holds if there is no iteration t , no M_1 and no M_2 that confirm the inequalities of Equations 3.9(a) or (b).

$$(M_1, M_2^*(q_2^t)) >_1 (M_1^*(q_1^t), M_2^*(q_2^t)) \quad \text{Eq. 3.9(a)}$$

$$(M_2^*(q_2^t), M_2) >_2 (M_1^*(q_1^t), M_2^*(q_2^t)) \quad \text{Eq. 3.9(b)}$$

When a SPE occurs, there is no phase of the game in which one of the players might want to alter its preference concerning the pair of FSM's in order to accomplish its next move. The distinction between the Nash and semi-perfect equilibria is that in the former, the concept of optimality is linked to the selection of the machines at the beginning of the game, while in the latter the preference about the FSM's endures throughout the game, before every iteration.

The solution of the game consists in finding a pair of machines for which the SPE holds. For all stages of the process, each FSM must satisfy the conditions:

- i. Neither player is able to increase its expected gain by unilaterally replacing its machine;
- ii. No player can reduce the number of internal states of its machine;

The approach of the IPD replacing conventional rational players by finite state machines is quite interesting, and has been giving margin to several further developments. In a subsequent paper, Abreu and Rubinstein [ABRE88] extended the analysis of the IPD played by FSM's, exploring aspects of Nash equilibrium and refining the notion of *machine complexity*, with its respective implications in the costs, gains and optimal machine selection.

II - Genetic Algorithms

Using the AI technique *Genetic Algorithms* (GA) developed by John Holland [HOLL75], Axelrod examined the IPD where the population of players was submitted to an evolutionary adaptive process. The GA method performs variation and selection of individuals, taken as possible candidates for an optimal solution of a problem. Under the GA approach the level of abstraction considered is the *Genotype*, usually represented by a binary coding.

In contrast to his previous tournaments, the restriction of fixed strategies was relaxed [AXEL87]. Each entry was denoted by a string with 70 positions (bits, or genes), that could be zero (DEFECT) or one (COOPERATE). The criterion for designating a generic strategy took into account all the possible combinations of the three previous moves, which

were employed to define the current action⁵⁰. The resulting total number of strategies is 2^{70} . This number is so huge, that it is intractable even to brute strength computing. Axelrod commented that, *"if a computer had examined those strategies at the rate of 100 per second since the beginning of the universe, less than one percent would have been checked by now."* So, the idea was to initialize a set of entries that would evolve by means of selection by relative fitness and generation of offspring by crossover and mutation. One of the main objectives of the experiment was to find out whether the GA could discover the TIT-FOR-TAT strategy [LEVY92].

Again, a computational simulation was programmed and implemented. The population of players to take part in the simulations comprised 20 random strategies, plus 8 other selected representatives, which were picked from the best performing strategies from Axelrod's second computational tournament⁵¹. A round of simulations involved 50 generations with 40 trials, each comprising 24000 iterations. Each game consisted of 151 moves per player, the same average from the original tournaments. The most successful strategies were *coupled* after all iterations in a given generation had been accomplished.

The criterion employed by Axelrod was to allow one coupling for the average rules (those inside the interval ± 1 standard deviation), two couplings to rules with upper punctuation (above 1 s.d.) and no couplings to the inferior strategies. Each *couple* generated two offspring, by means of the crossover GA operation. The other GA operator used was mutation (changing a C for a D and vice-versa)⁵².

Two different series of simulations were run. In the first, the 20 randomly selected strategies confronted only the eight mentioned representatives, which remained static. For this series, the flowchart of operations subsumes the following steps:

⁵⁰ The 70-position chromosome resulted from a string of 64 bits, which responded for the $4^3=64$ possible combinations derived from the four attainable outcomes (CC, CD, DC and DD) in the preceding three interactions, plus six additional bits regarding the player's move for the initial combinations of under three interactions. Thus there were 2^{70} possible strategies.

⁵¹ The second tournament (see section 3.3.2) had 62 entries, and the eight chosen rules have been considered as a reasonable account for how well a given strategy did with the entire set [FOGE94]. Their special feature was the "representativeness", under the view that the score that a particular rule would get when ran against all others was highly correlated with the gain that could be obtained with the eight representatives.

⁵² Crossover and mutation probabilities averaged one and one-half per generation, respectively.

1. Initialize a population of 20 randomly selected strategies;
2. Perform a round-robin tournament of each random strategy against the 8 special representatives and record the weighted average payoff;
3. Determine the number of offspring from each parent policy proportionally to their performance;
4. Create offspring by crossover and mutation;
5. Return to step 2.

By the end of the experiment, Axelrod noted that the GA method evolved populations whose median performance, in terms of accumulated gains⁵³, was about the same as TFT, with behaviors also resembling it.

The second sequence of experiments consisted of ten trials and required that the evolving policies played facing each other, now excluding the eight fixed rules. The complexity of this new environment greatly increased, because now the players confronted opponents that were also concurrently evolving. The highly competitive process meant that a rule might either match its adversaries' ever increasing efficacy or become extinct by the selection mechanism.

As a general outline, the population departed from initial cooperation, but inclined towards equal replication whenever a cooperative partner was recognized. The population's average score raised along the process, what meant that the matured participants in the game developed the skill of differentiating cooperators from defectors.

II - Evolutionary Programming

Another important contribution to the IPD as played by logical stimulus-response devices was provided by Fogel [FOGE93a]⁵⁴, [FOGE94a], [FOGE94b], [FOGE94c], [FOGE95], who studied the evolution of strategies in a population of FSM's. Fogel employed the AI technique known as evolutionary programming (EP) [FOGE66] to evolve the finite machines' behaviors along the iterations.

⁵³ The payoff matrix was the same employed in the original contests.

⁵⁴ An update of this paper is provided by Harrauld and Fogel [HARR95].

Like Genetic Algorithms (GA) and Evolution Strategies (ES)⁵⁵, Evolutionary Programming is one of the lines of investigation in the broader field of Evolutionary Computation. Given its characteristics, EP has been successfully applied to simulated evolution in games, operating directly on finite populations with varying mutation rates. On the other hand, ESS analysis requires infinite populations and no mutation, just ecological selection [SMIT82].

Regarding the Prisoner's Dilemma, EP, in contrast to the GA approach, does not use a mapping of the previous moves as a background for formulating the players' strategies. Instead, EP depicts the behavior of evolving individuals employing Finite State Machines (FSM's) [FOGE94c]. As a sequence to an earlier investigation [FOGE91], Fogel performed simulations of the IPD with FSM's according to the following procedure [FOGE93a]:

1. Initialize a population of 100 FSM's at random starting with 1 to 5 states, but being allowed to possess a maximum of 8 states⁵⁶ in the course of the process;
2. Perform a round-robin tournament with 151 iterations per player, what constitutes a *generation* (the maximum number of generations allowed per trial was 200);
3. Assign the respective payoffs to the competitors;
4. Evaluate fitness, in terms of the accumulated payoffs;
5. Select the 50 best ranked machines to become parents of an equal number of *offspring*, which replace the 50 machines with the lowest punctuation in the next generation;

⁵⁵ Developed by Schwefel [SCHW65] and Rechenberg [RECH73]. The essential differences between ES and EP are: (1) ES rely on strict deterministic selection. EP typically emphasizes the probabilistic nature of selection by conducting a stochastic tournament for survival at each generation. The probability that a particular trial solution will be maintained is made a function of its rank in the population. (2) ES typically abstracts coding structures as analogues of individuals. EP typically abstracts coding structures as analogues of distinct species (reproductive populations). Therefore, ES may use recombination operations to generate new trials [BACK93], but EP does not, as there is no sexual communication between species [FOGE93b].

⁵⁶ Fogel argues that "Eight-state FSM's do not subsume all the behaviors that could be generated under the coding of Axelrod [AXEL87], but do allow for a dependence on sequences of greater than third order." Furthermore, "this limit (of eight states) was chosen to provide a reasonable chance for explaining the behavior of any machine through examination of its structure." Op. cit [FOGE95], p. 209.

6. Apply *mutation* to the parents in order to originate the offspring — one offspring per parent. Mutation could be one of the six modes [FOGE66]:
 - Change an output symbol (C or D)⁵⁷;
 - Change a state transition;
 - Change start state;
 - Add a state;
 - Delete a state;
 - Change start symbol for the first move.
7. Go to step 2.

In that experiment, in despite of an initial tendency to mutual defection, after 5 to 10 generations, the evolution of the machines brought about the traditional reciprocating behavior, in which the players cooperate with equivalent policy holders and defect otherwise. Such an evolved environment turned out to be quite inhospit to defectors, which were not able to persist. Those conclusions follow the same pattern found by Axelrod [AXEL87].

To find out whether the population size affected the rate at which cooperative behavior evolved, Fogel accomplished another set of experiments, with variable population sizes: 50, 100, 250, 500 and 1000 parents.

Figure 3.3 shows a diagram of the best evolved FSM for a population of 50 parents.

⁵⁷ The input symbols are (CC), (CD), (DC) and (DD).

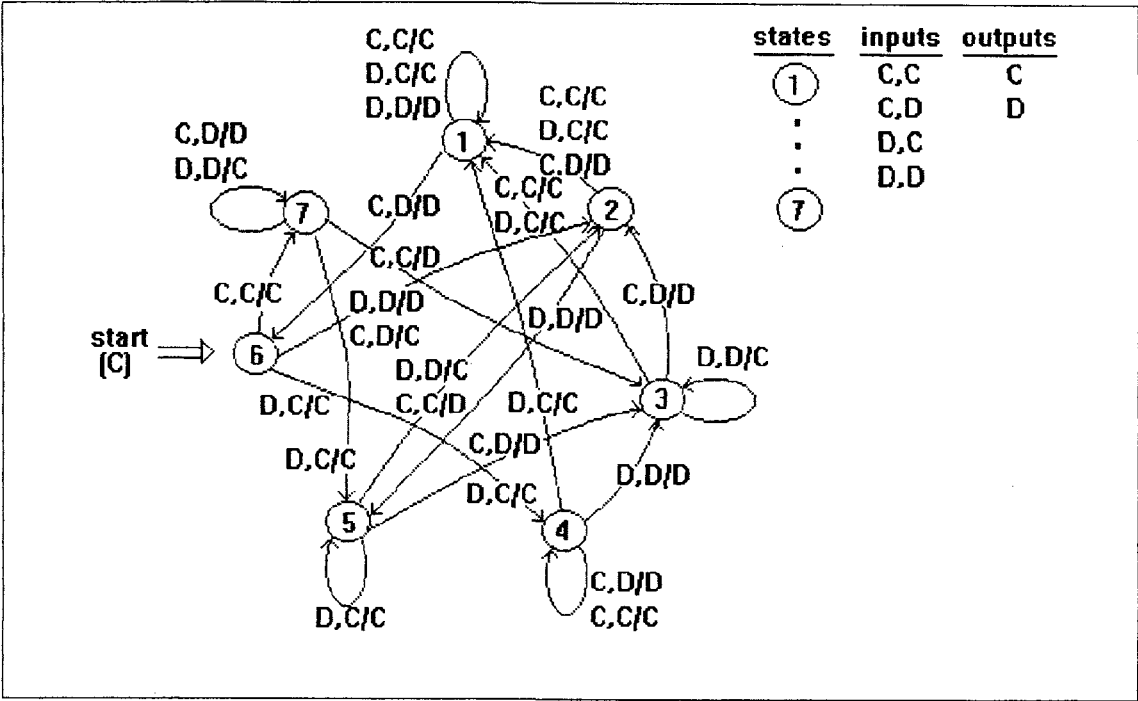


Figure 3.3 - Schematic Diagram of a Best Evolved FSM with Seven States

The IPD with FSM players utilized the traditional dichotomic form of the game, with only the discrete inputs and outputs composed exclusively of C's and D's. However, in a subsequent work, Fogel investigated a modified IPD with continuous actions between those two extremes, that is discussed next.

III - Evolutionary Simulation of the IPD with Neural Networks

In the IPD, a strategy can be considered a *transfer function*, which transforms a given sequence of inputs into an output. In the preceding case, the FSM's were an adequate representation for evolving strategies because of the discrete characteristic of both input and output. But if those variables are allowed to vary continuous within an interval, the problem of representing players by means of logical stimulus-response devices cannot be handled by automata with a finite number of states. To overcome that restriction, Multi-Layer Perceptrons⁵⁸ (MLP's) can provide an effective form of replicating the contestants.

⁵⁸ A *Perceptron* is a feedforward network with one output node (or *neuron*) that learns a separating hyperplane in a pattern space. The simplest model of the perceptron has two layers, one for the inputs (activations) and another for the outputs (signals). When hidden (intermediary) layers are added, the term *Multi-Layer Perceptron* is used [ROSE62], [ANDE88], [KOSK92].

Using this method, Fogel [FOGE94c] and Harrald [HARR95] performed simulations of the IPD. Each player's strategy was depicted by an MLP with 6 input nodes, a prescribed number of hidden nodes and a single output node. The output (t = current move) was determined having as basis the three preceding interactions ($t-1$, $t-2$, $t-3$). Figure 3.4 illustrates the Multi-Layer Perceptron employed⁵⁹.

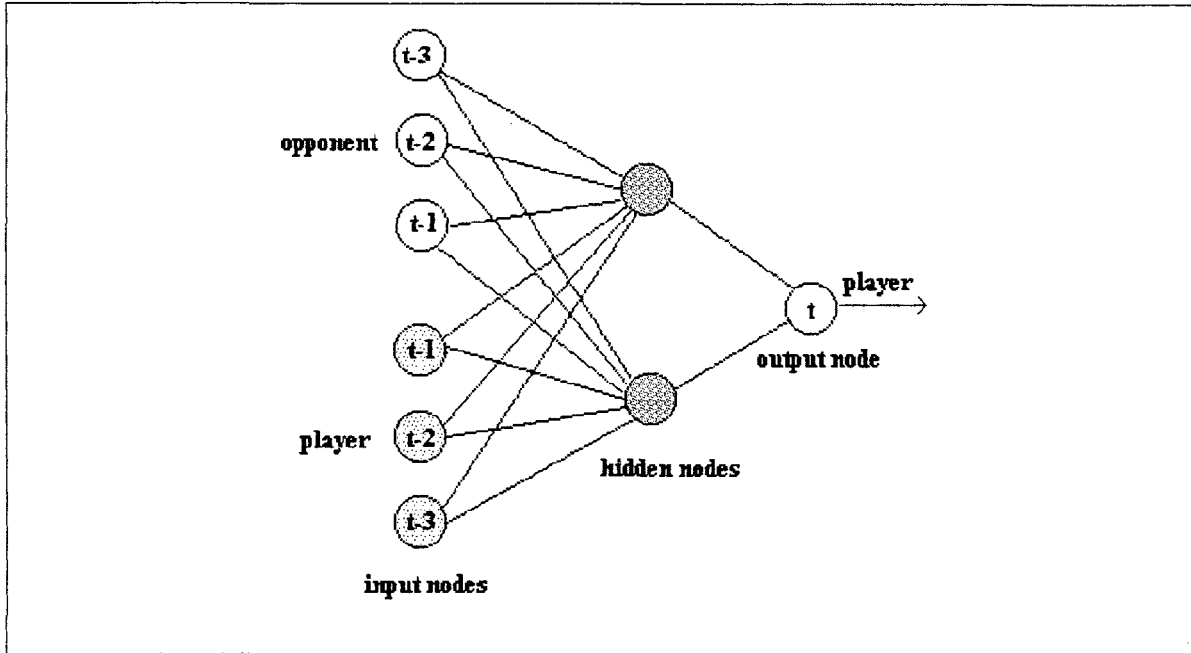


Figure 3.4 - A Multi-Layer Perceptron for the IPD with continuous actions and Payoffs

A planar approximation function supplied the payoffs, which varied in the interval $[-0.25, 4.75]$. The payoffs were calculated as a function of the players' pair of actions α (player A) and β (player B). α and β assumed values between -1 and 1, which correspond to full defection and full cooperation, respectively⁶⁰. The payoff to player A is supplied by Equation 3.10.

⁵⁹ From [FOGE94c].

⁶⁰ The nodes (or *neurons*) act as functions, transducing an input activation into an output signal, bounded to a specified interval. An example of a sigmoidal function is the differentiable logistic function

$$S(x) = \frac{1}{1 + e^{-cx}}, \text{ which approaches the } \textit{threshold function} \text{ (non-differentiable) when the constant } c$$

increases. $S(x)$ transduces negative activations to *zero* signals, and positive activations to the *unity* signal. *Zero* activation corresponds to the *threshold value*, that can be arbitrarily transduced to 1, 0 or the previous signal value. In Fogel's model, all nodes of the MLP used sigmoidal filters scaled to $[-1, 1]$. Op. cit. [KOSK92], p. 39-40.

$$f(\alpha, \beta) = -0.75\alpha + 1.75\beta + 2.25. \quad \text{Eq. 3.10}$$

The routine for the evolutionary simulation involved five main steps [FOGE94c]:

1. A population of a chosen number of networks (MLP's) was initialized at random, with all weights and biases starting uniformly distributed over $[-0.5, 0.5]$;
2. Each parent network generated an offspring, which was done by adding a standard gaussian random variable to every weight and bias term;
3. All networks played against each other in a round-robin tournament with 151 iterations per competitor, constituting a *generation*. The fitness of each MLP was taken as its average payoff per iteration;
4. After ranked according to their respective fitness, the top half was selected as new parents;
5. The procedure returned to step 2 until 500 generations had been executed.

Two sets of experiments were accomplished employing this method. The first used networks with 6 input neurons, 2 neurons in the hidden layer and one output node (6-2-1). The second framework had an increased complexity, with 20 nodes in the hidden layer, and the same input/output configuration (6-20-1). For each setting ten trials were conducted, with populations sizes of 10, 20, 30, and 50 parents.

At the end of the series of experiments, the following general results were achieved:

- The 6-2-1 MLP's did not show a tendency for cooperative behavior for any population size;
- When the population size was above a minimum value, a mutual cooperation trend was likely to develop in the 6-20-1 networks. In one of the trials (#4), the average score reached about 2.9 points for the (6-20-1) MLP with 50 parents.
- Even when a cooperative tendency arose, it did not incline towards complete cooperation.
- Although total and irreversible defection was generally associated with the simulations using the (6-2-1) networks, it was also likely to be acquired by the networks of the (6-20-1) setting.

The simulations provided several significant findings regarding the tradeoff between methodological features of the process employed and other broader aspects of the IPD. Some of the conclusions are listed below.

- Mutual cooperation required a minimum complexity for the players (MLP's), which is equivalent to the availability of flexible behaviors;
- The (6-20-1) networks were capable of reasonably sustainable cooperation, but not the (6-2-1) MLP's.
- The level of cooperation lacked both completeness and stability, with the parents' mean payoff peaking at about 3.0 and declining thereafter, as the number of generations grew. Total and unrestricted cooperation would have yielded 3.25.
- No level of cooperation seemed to show stability. The more steady evolved policy was mutual defection, with a very scarce tendency towards the restoration of cooperation.
- The population size appeared to affect the chance of the development of cooperative behavior in (6-20-1) networks, though quick instability in the mean payoff could be observed even in populations of 30 or 50 MLP's.

Fogel suggests further research on the subject. One point that remained unsettled, for example, was that the average payoff fluctuated between *nearly complete cooperation*, *neutrality* and *nearly complete defection* within one or only a few generations. He speculates about the reason why that happened, that could be the result of parents interacting with diverse offspring or simply caused by the replacement of the parents by their offspring.

3.5 - The One-sided Prisoner's Dilemma

The traditional PD is a symmetric game, in the sense that the players have the same available set of strategies and also the payoffs are equivalent, if the actions are reversed. However, there are other games with resembling properties in which an identical *dilemma* persists for only one of the players, which have been termed *one-sided Prisoner's Dilemmas* (OSPD) [RASM89]. The OSPD does not obey the specifications of the original PD, because neither the strategies nor the payoffs are symmetric. In the OSPD the

pair of actions (C, C) is strictly preferred by at least one of the participants, what departs from the usual favored (D, C). The *product quality game*⁶¹ is an example of an one-sided PD. Its simplest version is presented in Figure 3.5 in its normal form with arbitrarily selected numeric payoffs.

| | | Consumer | |
|--------|---------------------------|-------------------------|--------------------------|
| | | COOPERATE (C) (Buys) | DEFECT (D) (Boycotts) |
| Seller | COOPERATE (C) (Honest) | (5, 5) | (-5, 1) |
| | DEFECT (D) (Deceits) | (10, -5) | (0, 0) |

Figure 3.5 - Normal Form of the Product Quality Game - An one-sided PD

For the game depicted above, if it is played only once, it is easy to see that the Consumer would like that both parties cooperate (C, C), while the Seller values (D, C) most. The Seller has two options (strategies): Be *honest* (COOPERATE), offering a product of adequate quality, or *deceit* (DEFECT) by supplying an inferior article. The Consumer cannot discern the quality before the purchase is effectuated. If the information is symmetric, that is, if the Seller also does not know the Consumer's decision, an one-sided PD is present. It would be better for both if the Seller cooperated, but its Weakly Dominant strategy is *deceit*, hence the Consumer *boycotts*. Therefore, the OSPD possess (D, D) as both Nash and Iterated Dominant strategy equilibria, though not a Dominant equilibrium⁶².

In case the OSPD is iterated with a finite number of repetitions, the equilibria found in the one-shot version remains valid, as demonstrated by Selten in his paper "The Chainstore Paradox"⁶³ [SELT78]. On the other hand, if either infinite repetitions are

⁶¹ From op. cit. [RASM89], p. 95.

⁶² See Chapter 2 - *A Review of Fundamental Topics of Game Theory*

⁶³ In that paper, Selten discusses the occurrence of inconsistencies between logical reasoning (backward and forward induction), as prescribed by Game theory, and plausible human behavior. Selten states that "...in this game, well informed players are expected to disobey the theoretic normative recommendations."

allowed or a *reputation effect*⁶⁴ is to be considered, the Chainstore Paradox does not apply. In the former case, diverse equilibria can hold, as a consequence of the Folk Theorem⁶⁵.

The one-sided PD is specially important for this dissertation, because it is the basic paradigm adopted in the model developed in Chapter for the practical application of the Fuzzy IPD.

3.6 - Applications of PD-related Games to economic problems

Aside from other scientific areas of study already broached, the Prisoner's Dilemma is likewise quite pervasive in economic models. It is often considered *the archetypal choice problem*, because it contraries some basic tenets of liberal economics⁶⁶, and "...is arguably the most influential discovery in game theory since its inception." [POUN92].

In this section, although a comprehensive review is impracticable, given its extent, a few salient applications of the PD to economic problems are considered.

3.6.1 - International Trade Tariff Policy

If a country that has an important share in international commerce, it can manipulate prices in its own advantage by the adequate setting of imports and exports tariffs. But if two countries simultaneously decide to impose protective tariffs, the outcome may be unfavorable for both. For any participant in this dispute, the most preferred result is that it can establish high restrictions while the other keeps its tariffs low. This situation corresponds to a DEFECT-COOPERATE outcome in a traditional Prisoner's Dilemma. The next best desired outcomes are (C, C), (D, D) and (C, D), in that order. In this PD-like conflict of interest the equilibrium requirement would make the nations ending up with a mutual defection that is disadvantageous for both [MCM92]. It should be observed that

⁶⁴ See op. cit. [RASM89], p. 94-98.

⁶⁵ See op. cit. [RASM89], p. 91-93.

⁶⁶ Liberal doctrines claim that the group welfare is furthered by the independent pursuit of individual interests op. cit. [BACH77], p. 63.

even if agreements are reached, they are unable to be enforced by any authority, and, hence, self-enforcement must be present.

The self-enforcing mechanism, in that case, is provided by the rules of GATT⁶⁷ organization. They specify that whenever a participant departs from the contract and raises its rates, an immediate and proportional retaliation from the affected countries should be accomplished, which is done by also raising their own tariffs on the *defector's* exports.

Although the mentioned sanction procedure is sometimes seen as negative, it is in fact the only basis for supporting the cooperative agreements' abiding, therefore being in the collective interest of the treaty's signatories. One of GATT's goals is to make its regulations easily observed and understandable by its members, thereby assuring the *transparency* of the process. Accordingly, the retaliatory strategies in the iterated game are effective only if eventual deviations can be detected, which is ensured by clear norms⁶⁸.

3.6.2 - Share Takeover⁶⁹

Assume that a company is being mismanaged, and in those circumstances, a share is worth v . If the management team is dismissed, the share's value would raise to $(v+x)$, but no shareholder has enough shares to take that action alone. Suppose an outside bidder makes a tender offer, valid for any number of shares offered, but conditional upon obtaining a majority, that is, more than 50% of the shares. Any bid p per share between v and $(v+x)$ is attractive to both bidder and shareholders.

What might happen is that the shareholders will not accept the offer. This is so because the bidder would not make any profit if it bids more than $(v+x)$, therefore those are dominated strategies. On the other hand, an individual shareholder would be willing to hold out for the new $(v+x)$ value, hence refusing p ($p < (v+x)$). This constitutes a typical *free-rider* situation, where every participant is refusing to take part in the arrangement, but expects to enjoy the resulting advantages when the deal is accomplished, thanks to the efforts of others who accepted less.

⁶⁷ General Agreement on Tariffs and Trade.

⁶⁸ If some non-cooperative conducts can be concealed and put in practice, the equilibrium point of the game would tend towards the traditional defective solution

⁶⁹ This example was provided by Rasmusen, in op. cit. [RASM89], p. 301-302.

A Prisoner's Dilemma plight turns out in takeovers if the bidder employs a *two-tie* tender offer. For instance, suppose the firm's value is 30, and a monopolist bidder proposes 10 for 51% of the stock and 5 for the other 49%, conditional upon succeeding in obtaining the control of the company. If a shareholder does not tender, it may lose the opportunity of being included in the 51% who received 10, and will get only 5 afterwards. Sell, which stands for DEFECT, is a weakly dominant strategy, and all would profit refusing in block to accept the offer.

3.6.3 - An Oligopoly Game

For this instance, let us consider the simplest form of oligopoly, the *duopoly*, in which the number of participating sellers is two. In this environment, a similar product is offered in the market by both firms A and B. The *Cournot* model is assumed, in the sense that the competitors choose their quantities and the price is a function of the total number of units produced. The selling price of a firm affects the other regarding its market share.

A numerical example⁷⁰ is provided in order to illustrate the game, with the cost and market demand (market clearing price) functions given by Equations 3.11(a), (b) and 3.12.

$$c_A = 4 - q_A + q_A^2 \quad (\text{A's cost}) \quad \text{Eq. 3.11(a)}$$

$$c_B = 5 - q_B + q_B^2 \quad (\text{B's cost}) \quad \text{Eq. 3.11(b)}$$

$$p = 10 - 2(q_A + q_B) \quad (\text{market demand}) \quad \text{Eq. 3.12}$$

The strategies of the duopolist firms is independently select a quantity output⁷¹. The available outputs were arbitrarily chosen as 0.92, 0.94 and 1.17 for q_A and 0.41, 0.74 and 0.94 for q_B .

⁷⁰ The example was borrowed from op. cit. [BACH77], p. 66.

⁷¹ A continuous output would yield an infinite collection of strategies; To avoid this, the strategy set is discretised, which consists in an approximation that gets better as the number of possible strategies increase. Also, the *utility* of each player is taken as the monetary profit attained.

Using those values and Equations 3.11(a), (b) and 3.12, the payoff matrix of Figure 3.6 results.

| | | q_B | | |
|-------|------|----------------------------|----------------------------|----------------------------|
| | | 0.41 | 0.74 | 0.94 |
| q_A | 0.92 | (3.14, 1.06) ^{JM} | (2.53, 1.38) | (2.16, 1.26) |
| | 0.94 | (3.16, 1.04) | (2.54, 1.35) ^{EQ} | (2.17, 1.23) |
| | 1.17 | (3.21, 0.85) | (2.44, 1.01) | (1.84, 0.78) ^{EP} |

Figure 3.6 - Payoff Matrix of a Cournot Duopolistic Game

The cell marked EQ is the only Nash equilibrium pair, because neither player would want to unilaterally deviate from it, provided the other remains fixed. On the other hand, the pair JM stands for the *joint maximum* profit. Should a *collusion* be made possible, meaning an arrangement between the players that would split the total gain, both could be better off⁷². In this way, the top left sub-matrix (marked in bold contour) likens a PD⁷³, because the dominance principle impels the players to EQ.

The cell with the EP superscript corresponds to the efficient point, in the sense that it represents the combination of productions such that marginal cost equals price, and it is only possible by outside ruling.

The current example has assumed no reciprocal effects in the total production costs of each firm. Nonetheless, even if the quantities produced by one firm influences the costs of the other, a *quasi*-PD again may result, though with non-symmetrical payoffs⁷⁴.

⁷² The total profit in EQ is 3.89, whereas in JM it is 4.20; The additional total gain of 0.31 could be shared in some way that would be certainly advantageous for both duopolists.

⁷³ There is just a resemblance with a PD, not an identity, because B has only the prospect of reaching an effective agreement with A regarding the extra profits, and not a possible gain as those derived solely from the game's rules. In a PD, besides the dominance force, the cooperative outcome (mirrored by JM in this example), is actually Pareto-better (see chapter 2 of this dissertation).

⁷⁴ See op. cit. [BACH77], p. 74.

3.7 - Concluding Remarks

The existing and potential applications of the PD paradigm to scientific and practical problems are numerous. In many situations, the model may not be a real PD in the traditional sense, with its respective rules. But in most instances the essence of the same dilemma persists, with intuition defying rationality, as in the 2×2 games *Chicken* or *Stag Hunt*, briefly discussed in section 3.2.1. This is also the case of the *one-sided* PD (section 3.5) and of the economic applications featured in section 3.6. The potential is vast. For instance, Hardin [HARD82] provides a comprehensive account of the free rider and the provision of public goods, under a multi-person PD approach, and Rasmusen [RASM89] illustrates several problems regarding Industrial Organization using PD-like paradigms.

Under a more theoretical point-of-view, but still with a orientation towards the understanding of practical questions, Brams recently developed what he named “Theory of Moves” (TOM) [BRAM93], where the PD is employed as one point-of-reference and the addition of a dynamic dimension to classical game theory is proposed. TOM also focuses on interdependent strategic situations in which the outcome is conditional to the choices that all players make. Furthermore, it radically alters the rules of playing, enabling the participants in the game to look ahead —sometimes several steps— before making a move. Using TOM, in the PD there are two nonmyopic⁷⁵ equilibria (NME’s) which are (D, D) and (C, C). This angle of the PD was well explored in a paper co-authored by the author of this dissertation [KAND95].

One of the most important restrictions for the use of the original PD in modeling real-life problems is its dichotomic feature, with only two actions permitted. To overcome this difficulty, this author proposed a *Fuzzy* IPD, in which the game could be played with moves varying continuously in an interval whose extremes coincide with total cooperation and defection. In addition, the players would make their decisions based on *qualitative reasoning*, using fuzzy expert systems as a decision support system. Employing this method, some investigations regarding the robustness and effectiveness of acclaimed

⁷⁵ A nonmyopic equilibrium is associated with the players’ regard at the future consequences of their actions, including the impact that they might have on the opponents’ reactions [BRAM94].

strategies such as TFT and PAVLOV against other general rules were performed using a computational simulation program. The description and detailing of the whole procedure was made in a paper co-written by this author [BORG95] and is also presented as the Chapter 5 of the present Dissertation.

Chapter 4

A Review of Fuzzy Set Theory and Expert Systems

4.1- Introduction

The introduction of the notions pertaining fuzzy sets was first made by Lotfi Zadeh [ZADE65], and consisted in an extension of the previously existent concepts regarding the theory of crisp sets. As a cornerstone of fuzzy set theory is *fuzzy logic*. In the classical approach, a statement can only be either *true* or *false*, therefore assuming truth values of zero and one, respectively. Notwithstanding, in the realm of fuzzy logic the truth value attributed to any assertion may range continuously between these two extremes, thus having as its underlying foundation a multivalued logic [YAGE94]. For example, given a declaration P , it can be said that it has a truth value of p , which can be modified, in the same way as in the traditional theory, by the usual operators such as NOT, AND, OR, IMPLIES and IF AND ONLY IF [STEW93].

Fuzzy logic can represent a form of uncertainty and deals with qualitative characteristics of an object, which may be vague or lacking precision, such as *high*, *low*, *expensive*, etc. Some of its most common and successful applications regard decision or control problems where perceptions, intuitions, subjective judgments and adjustments are involved. Also, it is often the case that a system under analysis can only be described by nonlinear relations and very difficult to be mathematically modeled. In this area, too, fuzzy logic may prove to be a powerful tool, yielding quite satisfactory results.

In this chapter a review of the fundamental concepts of fuzzy logic, fuzzy measures and fuzzy expert systems are presented. The objective is to provide a background of the theory which is employed in other subsequent parts of the Dissertation, particularly in a

fuzzy approach of the Prisoner's Dilemma (Chapter 5) and in a model of this game directed to an application to a practical problem (Chapter 6).

The structure of the survey included in this chapter includes the following main topics:

- a) Fuzzy Measures, Belief and Fuzzy Set Theory;
- b) Operations on Fuzzy sets;
- c) Fuzzy Expert Systems.

The development of the theory regarding each of the mentioned topics has already reached a very advanced and specialized level, and the quantity of related technical material is vast. So, for the sake of objectiveness, only the paramount and specific aspects of the mentioned subjects are addressed. A list of bibliographic references is provided, where in-deep knowledge of the themes can be found.

4.2 - Fuzzy Measures and Fuzzy Set Theory

4.2.1 - General discussion

Since its introduction, along its development, and even presently, the fuzzy set theory has been criticized by many scientists and researchers. A large portion of the resistance against this field seems to derive from the idea that the results that can be achieved by means of modeling problems with the tools provided by fuzzy set theory may also be reached employing statistics and classical probability theory, and therefore there is no case for fuzziness¹. Nevertheless, lots of successful and efficient practical applications in a wide range of products and problems have demonstrated that the use of fuzzy sets can yield better results in many circumstances. Most certainly, there should be no reasons for such (sometimes) fierce dispute. Both fields of study have their merits and there are numerous specific areas of scientific interest where one approach is able to perform better than the other. The key proposition for the resolution of the controversy may lie in the fact

¹ Many critics claim that fuzziness can be replaced by subjective probability, which is usually defined by the degree of belief a person has that an event will occur [SMIT87]. This thought is a basis for Bayesian reasoning, whose adepts are some of the most critical of fuzzy set theory [STAL77], [LIND82].

that the concept of fuzzy sets represent a distinctive class of uncertainty that is *not* probabilistic in its essence. Furthermore, one has also to consider the notions of ambiguity, generality² and vagueness. Then, how should *fuzziness* be distinguished from those concepts? Zadeh [ZADE76], [ZADE79] claims that fuzziness is one component of vagueness. He gives an example using two statements³:

1. "Ruth has dark skin and owns a red Porsche."
2. "Ruth lives somewhere near Berkeley."

In statement 1. the adjectives "dark" and "red" are fuzzy because they have gradations, but they are not ambiguous. On the other hand, "somewhere near" in phrase 2. contains an ambiguity, since in this case, rather than gradation, the imprecision refers to Ruth's location. Following Zadeh's definition, vagueness is a combination of both fuzziness and ambiguity.

For clarity, consider a universe of discourse X and a subset $S \subset X$. The characteristic function $u_S(x)$ is a mapping $u_S: X \rightarrow [0, 1]$. If S is a crisp subset of X , $u_S(x)$ can only assume the values 0 or 1, meaning that x either *is* or *is not* a member of S . If S is a fuzzy subset of X , then $u_S(x)$ is usually called the *membership function* of S , and the values it assumes are *grades of membership* of an element x in S .

It is important to make a distinction between the *grade of membership* in a fuzzy set, that has unsharp boundaries, and the *degree of certainty* with which a given element belongs to a *crisp* set. The latter is termed a fuzzy measure, and its concept was first presented by Sugeno[SUGE77]. There are several classes of fuzzy measures, but they are always defined by a function which assigns to each subset of the universe of discourse X a number between zero and one.

² Also called nonspecificity.

³ Op. Cit. [SMIT87], p.11.

4.2.2 - Fuzzy Measures

A fuzzy measure is defined by a function $g: \mathbf{P}(X) \rightarrow [0,1]$, where $\mathbf{P}(X)$ is the power set of crisp subsets of the universe of discourse X^4 . The function g then maps every subset $A \in \mathbf{P}(X)$ in the interval $[0,1]$, and must obey the following three axioms of fuzzy measures⁵ [WANG92]:

- Axiom g1 (*boundary condition*): $g(0)=0$ and $g(X)=1$;
- Axiom g2 (*monotonicity*): $\forall A, B \in \mathbf{P}(X)$, if $A \subseteq B$, then $g(A) \leq g(B)$;
- Axiom g3 (*continuity*): For every sequence $(A_i, i \in \mathbb{N})$ of subsets of X , if either $A_1 \subseteq A_2 \subseteq \dots$ or $A_1 \supseteq A_2 \supseteq \dots$ (that is, the sequence is monotonic), then

$$\lim_{i \rightarrow \infty} g(A_i) = g \lim_{i \rightarrow \infty} (A_i)$$

The diagram shown in Figure 4.1, extracted from op. cit. [WANG92], illustrates the relationship among the main classes of Fuzzy Measures.

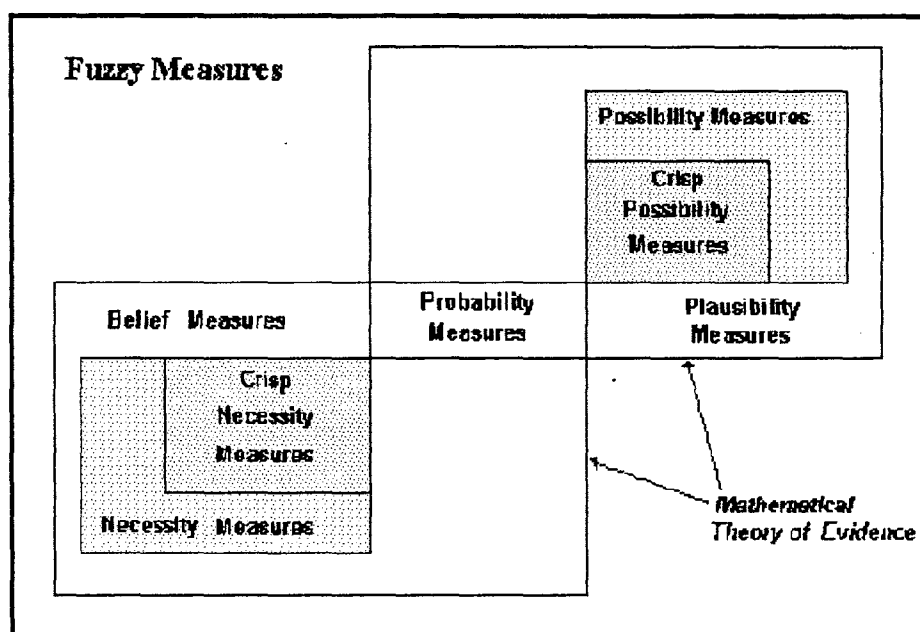


Figure 4.1 - Relationship among the Main Classes of Fuzzy Measures

⁴ The power set $\mathbf{P}(X)$ is the collection of all subsets of X , and can usually be replaced by a Borel field \mathbf{B} or sigma-field defined on X [KOSK92].

⁵ Axiom g1 means that independently of the amount of evidence, any element does not belong to the empty set and belongs to the universal set.

Axiom g2 states that the evidence of the membership of an element in a subset should be less or equal than the evidence of the membership of that element in a set which contains that subset.

Axiom g3 requires that $g(\cdot)$ be a continuous function.

In this part of the dissertation, the presentation will be restrained to some of the fundamental topics regarding Belief and Plausibility Measures, since those are the techniques that were adopted for the formulation of the model of application included in Chapter 6.

4.2.3 - Belief and Plausibility Measures

The application of Belief and Plausibility Measures has its origin in the Theory of Evidence, also called Dempster-Shafer Theory, after its founders [DEMP67], [SHAF76].

Both belief and plausibility measures consist of a mapping of the power set of X in the interval $[0, 1]$, that is:

$$Bel: \mathbf{P}(X) \rightarrow [0,1], \text{ and}$$

$$Pl: \mathbf{P}(X) \rightarrow [0,1].$$

Additionally to Axioms $g1$, $g2$ and $g3$, the following axioms for Bel and Pl must be satisfied, respectively, where A_1, A_2, \dots, A_n are subsets of A .

$$Bel(A_1 \cup A_2 \cup \dots \cup A_n) \geq \sum_i Bel(A_i) - \sum_{i < j} Bel(A_i \cap A_j) + \dots + (-1)^{n+1} Bel(A_1 \cap A_2 \cap \dots \cap A_n)$$

Eq. 4.1

$$Pl(A_1 \cap A_2 \cap \dots \cap A_n) \leq \sum_i Pl(A_i) - \sum_{i < j} Pl(A_i \cup A_j) + \dots + (-1)^{n+1} Pl(A_1 \cup A_2 \cup \dots \cup A_n)$$

Eq. 4.2

Making $n=2$, $A_1 = A$ and $A_2 = \bar{A}$ (the complement of A), Equations 4.1 and 4.2 yield the basic inequalities of belief and plausibility measures, respectively.

$$Bel(A) + Bel(\bar{A}) \leq 1; \quad \text{Eq. 4.3}$$

$$Pl(A) + Pl(\bar{A}) \geq 1. \quad \text{Eq. 4.4}$$

Belief and plausibility measures are associated in the manner depicted by Equations 4.5 and 4.6.

$$Pl(A) = 1 - Bel(\bar{A}) \quad \text{Eq. 4.5}$$

$$Bel(A) = 1 - Pl(\bar{A}) \quad \text{Eq. 4.6}$$

A function $m(\cdot)$, such that $m: P(X) \rightarrow [0,1]$, $m(\emptyset)=0$ and $\sum_{A \in P(X)} m(A) = 1$ can be

used to express $Bel(\cdot)$ and $Pl(\cdot)$. Considering the existence of uncertainty in the proposition that a particular element of X belongs to A , $m(A)$ stands for the degree of evidence that supports that claim, often called a *mass function* or a *basic probability assignment*. There is an important distinction between the meaning of $m(A)$, which is defined on subsets $A \in P(X)$, and that of probability distribution functions, defined on elements $x \in X$ ⁶.

It can be seen (see footnote 6) that $m(\cdot)$ does not satisfy the axioms of fuzzy measures, but it is used to determine $Bel(\cdot)$ and $Pl(\cdot)$.

$$Bel(A) = \sum_{B \subseteq A} m(B) \quad \text{Eq. 4.7}$$

$$Pl(A) = \sum_{B \cap A \neq \emptyset} m(B) \quad \text{Eq. 4.8}$$

The connection between $m(A)$ and $Bel(A)$ is that the former denotes the degree of evidence that a particular element $x \in X$ belongs precisely to a set A , while the latter designates the total belief that x belongs to A together with other specific subsets of A .

$Pl(A)$, on its turn, includes the amount of evidence assigned to $Bel(A)$ *plus* the belief that x belongs to other subsets whose intersections with A are not empty sets, that is, overlap with A . From these definitions it can be verified that the measure $Pl(\cdot)$ is less restrictive than $Bel(A)$, consequently entailing the relation $Pl(A) \geq Bel(A)$.

A set A is denominated a focal element of m if $m(A) > 0$. This definition means that the evidence denoted by m refers to A , its focus. A *body of evidence* is a list of the focal elements of m with their associated values $m(A)$.

The values $Bel(A)$ and $Pl(A)$ can be viewed as the lower and upper bounds of a probability interval assigned to a focal element $A \subset X$. A notable feature of the Theory of

⁶ The function $m(\cdot)$ has the following characteristics:

- It is possible that $m(X) \neq 1$;
- It is not necessary that $m(A) \leq m(B)$ if $A \subset B$;
- $m(A)$ and $m(\bar{A})$ may be unrelated.

Evidence is that it can denote degrees of ignorance about a proposition. From Equation 4.3, one can see that a certain amount of belief regarding a set A does not imply that the remaining belief should be assigned to the complement of A , \bar{A} [YEN90].

A very illustrative example of the use of Belief Functions is supplied by Thomas Strat [STRAT90]. This author uses the term *support* instead of *belief* to designate the lower bound of the probability interval⁷. The denomination *support* (Spt) is also adopted in the present work, notably regarding the elaboration of the decision process of the Consumers participating in the game described in Chapter 6.

Consider a Carnival Wheel with ten equivalent sectors, yielding the gain printed in each sector. Assume that there is a hidden sector, marked with a “?”, that can carry any of the other visible values. The problem is to compute an estimation of the prize that should be expected from such gambling device, and compare it with a fixed cost associated to the privilege of spinning the carnival wheel.

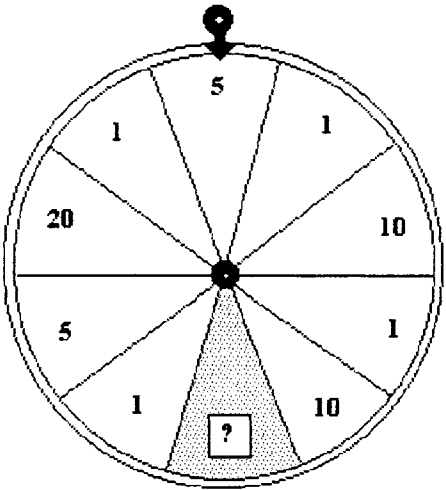


Figure 4.2 - Carnival Wheel with a hidden sector

The space X of possible outcomes is $\{1, 5, 10, 20\}$. The *mass* function, *support* and *plausibility* for the Carnival Wheel⁸ of Figure 4.2 are listed in Table 4.1.

⁷ The upper bound remains with the usual denomination, i. e., *plausibility* (Pl).
⁸ Note that the $\{1\}$, $\{5\}$, $\{10\}$, $\{20\}$ and $\{1, 5, 10, 20\}$ are treated as *focal elements*.

| | |
|-----------------------------|--|
| $m(\{1\}) = 0.4$ | $[Spt(\{1\}), Pl(\{1\})] = [0.4, 0.5]$ |
| $m(\{5\}) = 0.2$ | $[Spt(\{5\}), Pl(\{5\})] = [0.2, 0.3]$ |
| $m(\{10\}) = 0.2$ | $[Spt(\{10\}), Pl(\{10\})] = [0.2, 0.3]$ |
| $m(\{20\}) = 0.1$ | $[Spt(\{20\}), Pl(\{20\})] = [0.1, 0.2]$ |
| $m(\{1, 5, 10, 20\}) = 0.1$ | |

Table 4.1 - Mass function and Belief intervals for the Carnival Wheel of Figure 4.2

In respect to the hidden sector, there is a certain amount of ignorance about its real value. It is only known that it ranges from the most pessimistic number (in the view of the gambler who is paying for the privilege of spinning the wheel), 1, to the most optimistic, 20. Then, the *expected value interval* of the prize is defined as:

$$E(x) = [E_*(x), E^*(x)].$$

$E_*(x)$ and $E^*(x)$ are the lower and upper bounds of the interval, given by:

$$E_*(x) = \sum_{A_i \subseteq X} \inf(A_i) \times m(A_i)$$

$$E^*(x) = \sum_{A_i \subseteq X} \sup(A_i) \times m(A_i)^9$$

Computing $E(x)$ using the numerical values of the example results in $E(x) = [5.50, 7.40]$. If $E(x)$ is the parameter used by the gambler to make a decision on whether to accept or not the game, it can be said that the answer is definitely yes for a premium (the price of the bet) less than 5.50, and clearly no if it is greater than 7.40. However, what should be done if the premium lies in the interval $[5.50, 7.40]$?

Assuming that a probabilistic treatment shall be used to solve the problem, one could resource to the generalized insufficient reason principle [DUBO82]. This method proposes an equal distribution of the 0.1 probability assigned to the hidden sector among all its possible values, which would result in $E(x) = 6.30^{10}$.

⁹ $\inf(A_i)$ and $\sup(A_i)$ designate the smallest and largest element in the set $A_i \subseteq X$ (op. cit. [STRAT90]).

¹⁰ Another criteria could be assigning to each possible values the same probabilities observed in the visible sectors, in which case $E(x) = 6.0$.

Alternatively, Strat suggests an approach where the belief regarding either the pessimistic (1) or the optimistic outcome (20) are parametrized by an unknown probability p , ensuing:

$$\text{probability (hidden sector contains 20)} = p$$

$$\text{probability (hidden sector contains 1)} = 1 - p$$

Taking into account the known frequencies and the parameter p and performing a simple calculation, the expected value of the prize is $E(x) = 5.50 + 1.90p$. Using $E(x)$ as the decision criteria, the gambler would have to compare the premium with the prize that results from an estimated p . In the example, the feasible interval of the prize to be yielded from the carnival wheel corresponds to $p=0$ and $p=1$, that is $[5.50, 7.40]$. The two extreme values correspond to the *most pessimistic* and *most optimistic* assessment, respectively.

In the referred work (op. cit. [STRAT90]), the author proves that the expected utility $E(x)$ attained from a prospective interval is given by

$$E(x) = E_*(x) + p[E^*(x) - E_*(x)] \quad \text{Eq. 4.9}$$

where $E_*(x)$ and $E^*(x)$ are the *most pessimistic* and *most optimistic* appraisals, in that order, and p is the probability of the occurrence of the most favorable outcome.

According to this procedure, $E(x)$ has the role of a **distinguished point** within the feasible interval, bounded by $E_*(x)$ and $E^*(x)$ ¹¹.

The question now turns to the adequate selection of p . In the original approach, the probability p is supposed be obtained from previous data. In the model that is developed in Chapter 6 of the present work, the assessment of p ¹², though still based on the available evidential information, has a different connotation. It is interpreted as a parameter that the decision maker employs to predict the *degree of intensity* with which a future event will be in its favor. It is important to stress this distinction, since the method

¹¹ $E_*(x)$ and $E^*(x)$ are taken as the *Support* and *Plausibility* of x , respectively.

¹² In Chapter 6, the corresponding variable is termed λ_{ig}^y .

that is adopted to forecast p in the cited model does not rely on statistically significant data, but on incidental figures dynamically conveyed from the simulation process.

4.2.4 - Fuzzy Set Theory

The underlying concept of a fuzzy set is quite simple, yet intuitively appealing for the modeling of real world situations and problems. The advantages of the use of fuzzy sets are greatly amplified when one is trying to mirror the way an human decision maker reasons. For example, if a person is confronted with a purchase decision, it will most likely translate the present numeric variables to a qualitative scale, like *expensive*, *affordable*, *dependable*, etc, operate them in this environment and finally make the decision. If the item under consideration is indivisible, the solution will be dichotomous, whereas otherwise an arbitrated amount is the output of the process.

Given an universe of discourse X and a subset $S \subseteq X$, an element $x \in X$ may belong partially to S . This is equivalent to saying that the boundary of S is not sharp, therefore ensuing different possible degrees of membership of x in S . More specifically, S can be associated to a *characteristic function* which maps any element $x \in X$ in an interval $[0, 1]$, that is,

$$\mu_S: X \rightarrow [0, 1]$$

So, a fuzzy set S is described by a set of ordered pairs $\{x, \mu_S(x)\}$, where $\mu_S(x)$ is called the **membership function** of the element x in the fuzzy set S . When the universe of discourse X is the real line, $\mu_S(x)$ can be represented by a functional form. If X is a set of n discrete points x_i , $i = 1, 2, \dots, n$, $\mu_S(x_i)$ must be expressed by the set of pairs $\{ \mu_S(x_1)/x_1, \mu_S(x_2)/x_2, \dots, \mu_S(x_n)/x_n \}$, which is also a form of representation of the fuzzy set S . Figure 4.3 illustrates some common classes of membership functions.

The triangular and the trapezoidal shapes are by far the most customarily utilized forms of depicting a fuzzy set . One reason for that preference is the convenience of

designating $\mu_S(x_n)$ by linear functions, that, nevertheless can be modified by the introduction of linguistic hedges¹³. Furthermore, the form of the fuzzy set representing a certain domain should not be a crucial aspect of the technique, hence the widespread use of simple linear shapes¹⁴.

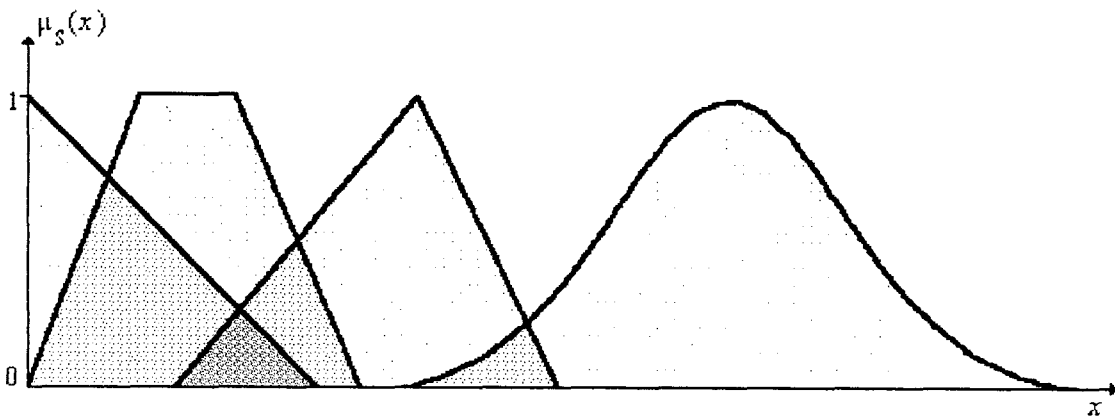


Figure 4.3 - Some shapes commonly employed for the membership function $\mu_S(x)$ representing a Fuzzy Set S.

Definitions

- ◇ If there exists at least one element x belonging to the universe of discourse X such that its membership grade in a fuzzy subset $B \subseteq X$ is equal to 1, then B is denominated *normal*; otherwise, B is called *subnormal*¹⁵.
- ◇ The largest membership grade of any element of a fuzzy set B is the *height* of B ; Therefore, if B is normal, its height is 1.
- ◇ Given a fuzzy subset B , all elements x whose grade of membership $\mu_B(x) \neq 0$ constitute a crisp set called the *support* of B , being denoted by

¹³ Linguistic hedges, or modifiers, operates on an original fuzzy qualification, expressed by an element's membership grade in a fuzzy set, and yields a new characterization for the element. See section 4.4.

¹⁴ Going still further, if a system is too sensitive to the form of the membership function at some point, maybe it should be the case to consider using an alternative process for the solution of the problem being modeled (Classroom notes from lectures on Fuzzy Sets given by Dr. A. Kandel, University of South Florida, Tampa, FL, Sept. 1994).

¹⁵ After op. cit. [YAGE94].

$$\text{Supp}(B) = [x \mid \mu_B(x) > 0, x \in X] \quad \text{Eq. 4.10}$$

- ◇ The *core* of a fuzzy subset $B \subseteq X$ is the crisp set of X whose elements obey the condition:

$$\text{Core}(B) = [x \mid \mu_B(x) > 1, x \in X] \quad \text{Eq. 4.11}$$

- ◇ Assuming B and C are two fuzzy subsets of X , if $\mu_B(x) \geq \mu_C(x) \forall x \in X$, then C is a subset of (or contained in) B , $C \subseteq B$. On the other hand, if $\mu_B(x) = \mu_C(x) \forall x \in X$, B and C are said to be *equal*. Consider, for example, $B = \{0.3/r, 0.2/s, 0.6/t\}$ and $C = \{0.2/r, 0.2/s, 0.5/t\}$. In this case, $C \subseteq B$, or C is a subset of B , but B is *not* a subset of C . The implication of this definition is that neither fuzzy subsethood nor supersetthood admits gradations, hence insinuating an accretion of the old black and white paradigm. This interesting question was addressed by Kosko [KOSK92], where he proposes an alternate approach, making use of the concept of a fuzzy set as a point in a hypercube.
- ◇ The power $M(A)$ of a fuzzy subset A is equivalent to its cardinality, sometimes called Σ -count (sigma-count), denoted by¹⁶:

$$M(A) = \sum_{i=1}^n \mu_A(x_i). \quad \text{Eq. 4.12}$$

Although the successful and efficient application of fuzzy set theory in several fields is an already established fact, a certain dispute between the latter and the classical probabilistic approach to uncertainty is expected to endure. Yet, some efforts have been aimed at reconciling these two concepts, in the sense of correctly identifying the situations

¹⁶ The cardinality, or power, of a fuzzy subset A must be distinguished from the **power set** of A , denoted by 2^A and defined as the set which contains all subsets of A .

where it is the case for using one or the other, as, for instance, the notions of fuzzy probability and fuzzy random variable, introduced by Kwakernaak [KWAK78]¹⁷.

4.3 - Operations with Fuzzy Sets

4.3.1 - Basic Concepts

◇ Union of A and B: $A \cup B = C$.

$$\mu_C(x) = \mu_A(x) \vee \mu_B(x) = \max [\mu_A(x), \mu_B(x)], \forall x \in X \quad \text{Eq. 4.13}$$

◇ Intersection of A and B: $A \cap B = D$. The intersection operator corresponds to the logical *and*, denoted by, such that

$$\mu_D(x) = \mu_A(x) \wedge \mu_B(x) = \min [\mu_A(x), \mu_B(x)], \forall x \in X. \quad \text{Eq. 4.14}$$

The union and intersection operations, performed by the max-min operators are of great importance in the fuzzy set theory, and have several distinguished properties. Some of them are listed below, where a, b and c are membership grades of an arbitrary element $x \in X$ in the fuzzy sets A, B and C, respectively.

- $\min (0, a) = 0$ for any $a \neq 0$;
- $\max (1, a) = 1$ for any $a \neq 1$;
- $\min (a, a) = \max (a, a)$ for any $a \neq 1$;¹⁸
- $\min (a, b) = \min(b, a)$; $\max (a, b) = \max (b, a)$; \Rightarrow Commutativity
- $\min (\min (a, b), c) = \min(a, \min (b, c)) = \min(a, b, c)$, and
- $\max (\max(a, b), c) = \max(a, \max (b, c)) = \max(a, b, c)$; \Rightarrow Associativity¹⁹

¹⁷ These notions are based on what is called *degree of realization* of a certain condition. For example, if fever is associated with influenza, then each condition (fever and influenza) can be described by an observer or expert as “moderately fulfilled”, “hardly fulfilled”, etc. So, one can calculate a fuzzy probability that a patient has a high fever given that the influenza is hardly realized [BECK95].

¹⁸ $A \cup A = A \cap A = A$

¹⁹ $A \cap (B \cap C) = (A \cap B) \cap C = A \cap B \cap C$; $A \cup (B \cup C) = (A \cup B) \cup C = A \cup B \cup C$.

- $\min(a, \max(b, c)) = \max(\min(a, b), \min(a, c))$, and
 - $\max(a, \min(b, c)) = \min(\max(a, b), \max(a, c))$; \Rightarrow Distributivity²⁰
- ◇ **Bounded sum:** $D = A \oplus B$, which means that the bounded sum of the fuzzy sets A and B is the fuzzy set D. The grade of membership in D is the sum of memberships in each fuzzy set A and B, with an upper limit of 1²¹:

$$\mu_D(x) = \min(\mu_A(x) + \mu_B(x), 1) \quad \text{Eq. 4.15}$$

If A and B are crisp sets, than $A \oplus B$ would be reduced to $A \cup B$.

Figure 4.4 illustrates the behavior of the min, max and bounded sum operators.

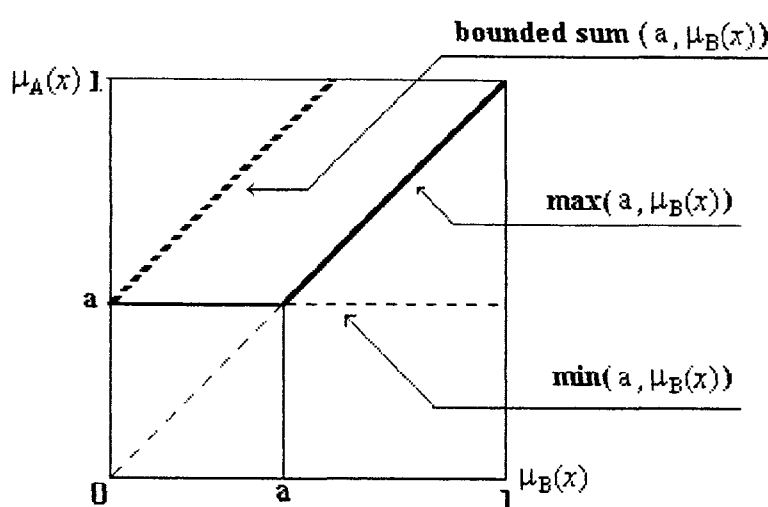


Figure 4.4 - Comparison of min, max and bounded sum operators

- ◇ **Negation (or complement):** $\text{not-}A = \bar{A} = X - A$;

$$\mu_{\bar{A}}(x) = 1 - \mu_A(x) \quad \text{Eq. 4.16}$$

²⁰ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$; $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

²¹ Smith (op.cit. [SMIT87]) designates the bounded sum, as defined, as an union operator, but he also considers the *intersection* or *and*-bounded sum, which is defined as $\mu_D(x) = \max(0, \mu_A(x) + \mu_B(x) - 1)$. In the latter case, if A and B are crisp sets, $D = A \cap B$.

◇ **De Morgan's Law:**

$$\max (\mu_A(x), \mu_B(x)) = 1 - \min ((1 - \mu_A(x)), (1 - \mu_B(x)))^{22} \quad \text{Eq. 4.17}$$

- ◇ **Product operators:** analog to those defined for joint probabilities of two independent events. For two fuzzy subsets A and B, comes:

$$\text{Intersection:} \quad A \cap B = \mu_A(x_i) * \mu_B(x_i) \quad \text{Eq. 4.18(a)}$$

$$\text{Union:} \quad A \cup B = \mu_A(x_i) * \mu_B(x_i) - \mu_A(x_i) * \mu_B(x_i) \quad \text{Eq. 4.18(b)}$$

- ◇ **α -level set:** Given a fuzzy subset $A \subset X$, the α -level set of A (also called α -cut), denoted by A_α , is a crisp set containing all elements x whose membership grade in A are equal or greater than α , $\alpha \in [0, 1]^{23}$. Therefore,

$$A_\alpha = \{x \mid (\mu_A(x) \geq \alpha, \forall x \in X)\}. \quad \text{Eq. 4.19}$$

With the concept of an α -level set, it is possible to represent any fuzzy set by means of crisp sets corresponding to its α -cuts [WANG92], [YAGE94].

Defining a fuzzy subset $F = \alpha A$ such that

$$\mu_F(x) = \mu_{\alpha A}(x) = \begin{cases} \alpha & \text{if } x \in F \\ 0 & \text{if } x \notin F \end{cases} \quad \text{for any } \alpha [0,1].$$

then

$$A = \bigcup_{\alpha \in [0,1]} \alpha A_\alpha$$

Example:

²² The following properties also hold under De Morgan's Law: $\text{not}(A \cup B) = (\text{not-}A) \cap (\text{not-}B)$; $\text{not}(A \cap B) = (\text{not-}A) \cup (\text{not-}B)$.

²³ A α -strong cut is described by $A_{\alpha+} = \{x \mid (\mu_A(x) > \alpha, \forall x \in X)\}$.

$$\alpha = 0.4$$

$$X = \{p, q, r, s, t\}; A = \{0.1/p, 0.3/q, 1/r, 0.6/s, 0.9/t\};$$

$$\alpha A = F = \{0/p, 0/q, 1/r, 0.6/s, 0.9/t\} \text{ (Fuzzy subset);}$$

$$A_\alpha = \{r, s, t\} \text{ (Crisp subset).}$$

If α varies, assuming its possible values in $[0,1]$, the union of all crisp sets αA_α will then result in the fuzzy set A .

4.3.2 - The Extension Principle

The extension principle is an important notion of the Fuzzy Set Theory. It provides a rule for associating the elements of two or more fuzzy subsets.

Suppose X and Y are crisp sets, A and B two fuzzy sets and f is a function such that $f: X \rightarrow Y$, and $f(x) = y$, $x \in X$, $y \in Y$. If $A \subset X$, $f(A)$ is a fuzzy subset of Y with the property

$$f(A) = \bigcup_x \{(\mu_A(x)/f(x))\}. \quad \text{Eq. 4.20}$$

The basic concept that underlies the extension principle is a *supremum of pairwise minima* [KOSK92]²⁴. In a more general and formal definition, the extension principle performs a mapping: $f: F(2^X) \rightarrow F(2^Y)$, where $F(2^X)$, $F(2^Y)$ are the fuzzy power sets²⁵ of X and Y , respectively.

Example:

Assume $X = \{\$1, \$4, \$9\}$ (prices of an item), and $Y = \{\text{VERY LOW, LOW, MEDIUM, HIGH, VERY HIGH}\}$ (perceived characteristics of the price of an item). Suppose that the function f is defined as:

$$f(1) = \text{LOW}$$

²⁴ In this work, Kosko criticizes the extension principle, arguing it is complicated, "... with the tendency to push fuzzy theory into largely inaccessible regions of abstract mathematics...", and "... achieves generality at the price of triviality." (op. cit., p. 300).

²⁵ The sets of all fuzzy subsets of X and Y , respectively.

$$f(4) = \text{MEDIUM}$$

$$f(9) = \text{VERY HIGH}$$

subset $A = \{ 0.6/1, 0.8/4, 0.1/9 \}$. Then,

$$f(A) = \{0.6/ \text{LOW}\} \cup \{0.8/ \text{MEDIUM}\} \cup \{0.1/ \text{VERY HIGH}\}$$

$$f(A) = \{0.6/ \text{LOW}, 0.8/ \text{MEDIUM}, 0.1/ \text{VERY HIGH}\}$$

Denominating $B = f(A)$, then B is a fuzzy subset of Y , such that $\forall y \in Y$,

$$\mu_B(y) = \max_{\text{all } x: f(x)=y} [\mu_A(x)].$$

4.3.3 - Triangular Norms and Co-Norms

Triangular norms (t-norms) and co-norms (t-conorms) are operators T and S that perform an aggregation of fuzzy subsets, so that they induce a mapping from a pair of membership grades into another, at this time regarding the fuzzy set that was originated from the respective aggregation operation.

$$T: [0, 1] \times [0, 1] \rightarrow [0, 1] \text{ (t-norm)}$$

$$S: [0, 1] \times [0, 1] \rightarrow [0, 1] \text{ (t-conorm)}$$

The min and the intersection product operators are examples t-norms, while the max, union product and bounded sum are instances of t-co-norms.

Given a and b , two arbitrary membership grades in two different fuzzy sets, a t-norm $T(a, b)$ operator must have the properties:

- $T(a, b) = T(b, a)$ (Commutativity);
- $T(a, T(b, c)) = T(T(a, b), c)$ (Associativity);
- $T(a, b) \leq T(a, c)$ when $b \leq c$ (Monotonicity);
- $T(a, 1) = 1$ (One Identity).

A t-conorm (also called s-norm), denoted by $S(a, b)$, shares identical properties with the t-norm, with the exception of the *One-Identity*, that is replaced by the *Zero-Identity*, so that $S(a, 1) = 0$.

The conditions of distributivity and idempotency are not present in t-norms and s-norms. Other pairs of $T(\cdot)$ and $S(\cdot)$ operators are cited in the literature, such as the Log operators²⁶, the Lorentzian operators²⁷ and the Yager operators²⁸ [SMIT87].

4.3.4 - The Fuzzy Integral

The fuzzy integral is intimately related to the concept of fuzzy measures, presented in section 4.2.2. It was introduced by Sugeno [SUGE77] is a convenient way of aggregating multiple quality factors of an object, with the objective of yielding a single synthetic evaluation index [TAHA90], [WANG92] [YAGE93].

Suppose $V = \{v_1, v_2, \dots, v_n\}$ is a finite set of factors regarding all the aspects, or *qualities*, of an object under appraisal. V is the *factor space* of the object. Now, the global evaluation is supposed to be reached taking into consideration the assessment of each aspect of the factor space V , which are *not* equally important. One of the most common solutions for this kind of problems is the weighted mean, which is based on the assumption that the factors are independent and therefore additive.

If a real number $g(E)$ in the interval $[0, 1]$ is associated to each subset $E \subseteq V$, indicating its importance in the global evaluation, this number should reflect the best judgment that could be made of the item considering only the factors that are present in the subset E . The function $g(\cdot)$ obeys the conditions $g(\emptyset)=0$, $g(V)=1$ and if $E \subset F \subset V$, then $g(E) \leq g(F)$ ²⁹, and is a fuzzy measure on $(V, 2^V)$.

²⁶ $T(a, b) = \log_s[1 + ((s^a - 1)(s^b - 1))/(s - 1)]$; $S(a, b) = 1 - \log_s[1 + ((s^{1-a} - 1)(s^{1-b} - 1))/(s - 1)]$.

²⁷ $T(a, b) = ab/(2 - a - b + ab)$; $S(a, b) = (a + b)/(1 + ab)$.

²⁸ $T(a, b) = 1 - \min[1, [(1-a)^t + (1-b)^t]^{1/t}]$, $S(a, b) = \min[1, (a^t + b^t)^{1/t}]$, for $t \geq 1$. In the Yager operator, when $t=1$, it becomes the **bounded sum** pair (see footnote 21) and when $t \rightarrow \infty$, it transforms into the **min-max** pair.

²⁹ \emptyset is the empty set; In Chapter 6 the functions $g(\cdot)$ and $h(\cdot)$ are designated $\phi(\cdot)$ and $\alpha(\cdot)$, respectively.

In the object's judgment process, for each of the factors $v_i \in V$, a *score* $h(v_i) \in [0,1]$ is settled by the examiner. Then, $g(v_i)$ and $h(v_i)$ are compounded by means of a fuzzy integral, denoted by $\int_F h \partial g$ ³⁰, corresponding to a *synthetic evaluation* of the item.

$$\int_F h \partial g = \max_{E \subset V} [\min (\min(h(v)), g(E))] \quad \text{Eq. 4.21} \quad v \in E$$

A good explanation for the workings of the fuzzy integral is given by Yager [YAGE94]. The function $g(E)$ is a measure of how the subset of factors E fulfills the *importance* concept, characterized by g . Note that $g(E)$ should be estimated independently and ahead of the examination of the specific object under appraisal, therefore consisting in a widely accredited postulation. On the other hand, $h(v)$ regards the degree of satisfaction of a particular subset of qualities pertaining the object. Thus, $h(v)$ corresponds to the *score* to be attributed to that subset. The operation $\min h(v)$ over all the elements $v \in E$ supplies the extent to which these elements satisfy h . The portion $[\min (\min(h(v)), g(E))]$ of the fuzzy integral matches the degree to which all $v \in E$ carry out the demands brought into the process by the functions g and h simultaneously. Finally, the \max of all minima concerning the subsets E is taken, yielding what is called the object's *synthetic evaluation*.

The fuzzy integral plays an important role in this dissertation, since it was the method of aggregation chosen to compound the factors concerning the judgment of an item by a Consumer, according to the model described in Chapter 6. For this reason, an illustrative example³¹ depicting the *modus operandi* of the fuzzy integral is presented.

Example

Suppose a potential buyer is evaluating a certain model of a car, and the features that are being taken into consideration in the analysis are *performance* (P), *comfort* (C) and

³⁰ In Chapter 6, the fuzzy integral is depicted by $\int_F \alpha \partial \phi$.

³¹ The example was inspired and is an adaptation of the problem of "evaluating a Chinese dish", in op. cit [WANG92], p.185.

price (M). Hence, these are the *qualities* or *factors* v_i , $i=1, 2, 3$, of the object, the car, and $V = \{P, C, M\}$. The set function $g(\{.\})$ appoints an importance measure to the subsets that result from the possible combinations that can be formed employing the individual factors P, C and M. The arbitrated values of $g(.)$ for the example are:

$$g(\{P\}) = 0.2$$

$$g(\{P, C\}) = 0.3$$

$$g(\{C\}) = 0.0$$

$$g(\{P, M\}) = 0.9$$

$$g(\{M\}) = 0.6$$

$$g(\{C, M\}) = 0.7$$

$$\text{and } g(\{V\}) = 1$$

The values of $g(\{.\})$ do not have the additive property, that is present in the traditional weighted mean method. The explanation for the assignment of *zero* to $g(\{C\})$ is that the factor *comfort*, taken isolated, does not have any importance in the item's appraisal. Conversely, another peculiar aspect is that the factors can exhibit a mutual reinforcement effect when taken together, as shown by the pairs $\{P, C\}$, $\{P, M\}$, $\{C, M\}$.

Then, to each factor, a score $h(v_i)$ is adjudicated, assumed as

$$h(P) = 0.4$$

$$h(C) = 0.6$$

$$h(M) = 0.7.$$

$\min h(v_i) = h(P) = 0.4$; $\min h(v_i)$ works as the parameter α of an α -level subset E_α of V , such that $E_\alpha = \{v \mid h(v_i) \geq \alpha\}$. In other words, E_α is the collection of importance factors whose scores are greater than or equal to $\min h(v_i)$, which obviously correspond to the whole set of importance factors, V . Therefore, $g(E_{\alpha=0.4}) = g(V) = 1$.

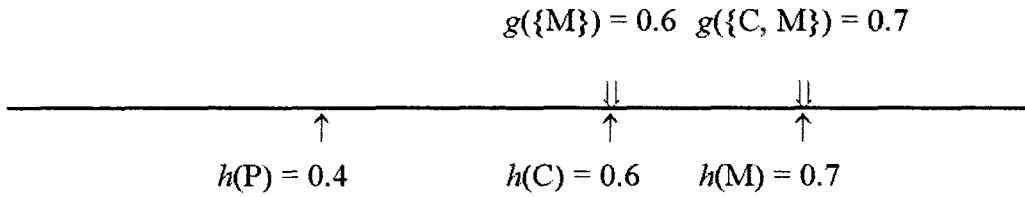
The other values of $g(E_\alpha)$ for $\alpha \geq \min h(v_i)$ are determined in a similar way, so $E_{\alpha=0.6} = V - P = \{C, M\}$; $g(E_{\alpha=0.6}) = g(C, M) = 0.7$ and $g(M) = 0.6$.

$$\int_E h \partial g = \max [\min[0.4, 1], \min[0.6, 0.7], \min[0.7, 0.6]]$$

$\int_{\mathbb{F}} h \partial g = 0.6$, which stands for the overall rate, or *synthetic evaluation* of that particular car³².

4.3.5 - Modification of the Fuzzy Integral

Regarding the previous example, a remarkable characteristic of the basic fuzzy integral can be noticed: The resulting value of the synthetic evaluation is not affected by some particular changes in the scores. To advocate this discussion, let us consider an horizontal axis where the significant values of the variables used for the computation of the fuzzy integral are plotted.



The synthetic evaluation of the car in the example that has been yielded by the fuzzy integral is $\int_{\mathbb{F}} h \partial g = 0.6$, which may have been originated from either $[h(C) \wedge g(\{C, M\})] = \min[0.6, 0.7]$, or $[h(M) \wedge g(\{M\})] = \min[0.7, 0.6]$. The values of $g(\cdot)$ are fixed for the evaluator and independent of the assignment of scores to the object under appraisal. Hence, the variable that is actually governing the synthetic evaluation for the range set of the example is $h(C) = 0.6$, because neither $h(P)$ nor $h(M)$ are having any influence whatsoever on $\int_{\mathbb{F}} h \partial g$ until the benchmark established by $[h(C) \wedge g(\{C, M\})] = 0.6$ is surpassed.

But it appears that, although the overall influence of each score in the computation of the fuzzy integral should still remain bounded by its respective befitting importance

³² The fuzzy integral is only one of a number of several techniques for multicriteria decision making. A particular mention is made to the method of *Ordered Weight Averaging* (OWA) [YAGE94], [YAGE95].

factor, an adjustment in $\int_F h \partial g$ should be promoted to contemplate some variations in the attributes $h(C)$. For instance, suppose that in the example presented, the value of $h(P)$ has assumed a different value. In this case, $h(P)$ could lie anywhere between zero and $h(C) = 0.6$, without having any impact on $\int_F h \partial g = 0.6$. Therefore, if an evaluator relies solely on the value yielded by the original fuzzy integral, it would be completely indifferent, in the situation depicted, to two items with scores regarding those extreme points $[0, 0.6]$. In order to overpass this intuitively undue feature of the original fuzzy integral, a correction composed of an increment (Δ^+) and a decrement (Δ^-) is proposed. The reasoning concerning this correction is as follows:

1. As the fuzzy integral is defined as a maximum operation over two or more minima, which on their turn are always represented by intersections of two fuzzy measures (score and importance), the final value yielded by $\int_F h \partial g$ must be necessarily equal to one of those minima, which is henceforth designated as the *governing value* of $\int_F h \partial g$ ³³.
2. The variations of the scores concerning the non-governing measures, which originally do not affect $\int_F h \partial g$ if their associated minima are less than the governing value, will be now accounted for. The ranges of relevant variations are always restrained by (a) the governing value ($h^\diamond(.)$ or $g^\diamond(.)$), (b) the other scores ($h(.)$) and (c) the importance factor $g(.)$ that is associated with the score under consideration.
3. Each relevant variation may consist of either an increment or decrement over $\int_F h \partial g$, depending on whether it is greater or lesser than the governing value, and taken as the difference between it and the nearest of (a), (b) or (c) described above³⁴.

³³ If there are two or more equal minima, each of them is separately picked as the *governing value*.

³⁴ In other words, the absolute value of the relevant variation to be accounted for is the minimum of the absolute values of the differences between the attribute under consideration and each one of (a), (b) or (c).

4. The increment (Δ^+) and the decrement (Δ^-) to be added to $\int_F h \partial g$ results from weighting each relevant variation to a function of the importance factor isolatedly associated to it³⁵.
5. The final synthetic evaluation of the item shall then be given by $\int_F h \partial g + \Delta^+ + \Delta^-$.

Example:

Taking the governing value of $\int_F h \partial g$ as $[h(C) \wedge g(\{C, M\})] = \min[0.6, 0.7] = 0.6$,

$$(a) = h(C) = 0.6;$$

$$(b) = \{ h(P) = 0.4; h(M) = 0.7 \};$$

$$(c) = g(C, M) = 0.7 \text{ and } g(P) = g(V) = 1 \text{ (because } h(P) = \min(h(P), h(C), h(M))$$

and the weights as $g(.)$ themselves, comes:

$$\Delta^+ (M) = g(M) \times \max[0, (h(M) - \max(h(C), g(M)))] = 0.6 \times (0.7 - 0.6) = 0.06;$$

$$\Delta^- (C) = g(C) \times \min[0, (h(C) - \min(h(C), g(C)))] = 0.7 \times (0.6 - 0.6) = 0.0;$$

$$\Delta^- (P) = g(P) \times \min[0, (h(C) - \min(h(C), g(P)))] = 0.2 \times (0.6 - 0.6) = 0.0;$$

Then, the final synthetic evaluation of the car of the example would be

$$\int_F h \partial g + \Delta^+ (M) = 0.6 + 0.06 = 0.66.$$

Discussion:

The increment $\Delta^+ (M) = 0.06$ resulting from the method proposed above tries to mirror an extra advantage that the evaluator is assigning to the car due to its score of Price $h(M)$. That gain represents the difference 0.1 relative the actual score and an inferior one, (which for the present purpose is limited to the governing value of 0.6), that could have

³⁵ The function can be the importance factor itself. In the Chapter 6, this method has been applied using the weights $g(.)$ for the increments (gains) and $g(.)^{\frac{1}{2}}$ for the decrements (losses).

been assigned to $h(M)$ without interfering with the basic $\int_{\mathbb{F}} h \partial g = 0.6$. It has weighted by $g(M) = 0.6$, thus yielding 0.06.

Considering that the original fuzzy integral consists only of minima and a maximum operations, its possible values must necessarily be one of the intervening variables $h(.)$ $g(.)$. With the proposed method, an hybrid approach is performed, with the convenience of making intermediary results possible. This system of performing a synthetic evaluation of an item with fuzzy measures will be employed in the application of a one-sided fuzzy iterated Prisoner's Dilemma presented in the Chapter 6 of this Dissertation.

4.3.6 - Linguistic Hedges

The imprecise terms usually employed to denominate fuzzy sets, such as HIGH, LOW, ATTRACTIVE, etc., can also undergo alterations by linguistic modifiers or *hedges*.

Formally, a linguistic hedge ℓ operates on an original fuzzy qualification Q_i , accomplishing a new characterization $R_i = \ell(Q_i)$. For example, the label "expensive" describing the price of an item can be reinforced by the hedge "very"³⁶. The resulting concept "very expensive" is still fuzzy, but with a different mapping rule from the numerical value of the price to the interval $[0, 1]$. Conversely, one can arrive at the notion "moderately expensive" using a symmetric line of reasoning. In this latter case, the hedge "moderately"³⁷ functions as a weakening modifier [ZADE72]. The new membership grade of an element of the universe of discourse in an fuzzy set altered by a linguistic hedge is generally expressed by a power of the original grade. The exponents 2 and $\frac{1}{2}$ are usually employed to represent "very" and "moderately", respectively. In this way, $\mu_{R_i}(x) = [\mu_{Q_i}(x)]^2$ for $\ell = \text{"very"}$ and $\mu_{R_i}(x) = [\mu_{Q_i}(x)]^{\frac{1}{2}}$ for $\ell = \text{"moderately"}$. The graphical representation of a standard fuzzy set and those resulting from the applications of those linguistic hedges is shown in figure 4.5.

³⁶ In the application model detailed in the Chapter 6 of this work, the term "very", rather than a linguistic modifier, is utilized to design an autonomous fuzzy set, when attached to other adjectival denominations, such as high or sensitive, for instance.

³⁷ Also equivalent to "more or less" or "sort of".

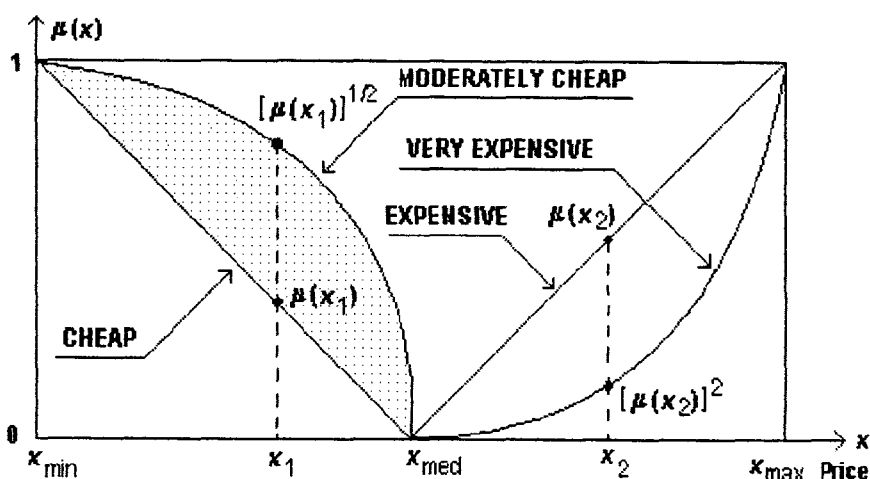


Figure 4.5 Example of Fuzzy Sets that result from the Application of Linguistic Hedges

Linguistic hedges can have basically two kinds of effect on the original concept: **Concentration**, like the modifier “very”, which brings about an *intensification* or *reinforcement* of the foregoing trait, and **dilation**, that consigns an *expansion* or *weakening* of the object’s primary feature. The generalization of the concentration and dilation operators admits any exponents other than the original 2 and 1/2. When the exponent is greater than 1, a concentration effect betides, and smaller membership degrees result. On the other hand, an exponent smaller than 1 implies in a dilation of the basal qualification, which produces greater membership values. The points x_2 and x_1 in Figure 4.5 illustrate the former and latter operators, in that order.

An elaboration of the idea of expanding and contracting basic fuzzy sets has been adopted in Chapter 6. There, a Consumer’s perceptions regarding the price and the quality of an item are affected by linguistic-like operators that act on the fuzzy sets employed to qualify these attributes, in the form of exponents taken as functions of the specific characteristics of an individual.

In addition to the better known concentration and dilation hedges, Zadeh [ZADE72] also explored contrast intensification and contrast diffusion modifiers. In the former, a fuzzy set is made less fuzzy by increasing membership values above 0.5 and

decreasing those below 0.5, thereby moving all membership degrees closer to either zero or one. In the latter, the membership values are headed for 0.5, and in this case boosting the fuzziness of the set [SMIT87].

In theory, there are no formal rules defining the exact quantitative influence of linguistic modifiers over fuzzy sets, and it can be said that the assignment of a numeric value to an exponent associated to a hedge is a quite subjective matter. Table 4 [COX95] lists a sample of feasible hedges.

| | | |
|---------------|--------------|------------|
| ESSENTIALLY | VIRTUALLY | VERY |
| SORT OF | RATHER | NOT |
| TECHNICALLY | ALMOST | MUCH |
| KIND OF | MOSTLY | PRETTY |
| ACTUALLY | ROUGHLY | ABOUT |
| REASONABLY | DEFINITELY | SOMEWHAT |
| APPROXIMATELY | MORE OR LESS | OFTEN |
| MODERATELY | NOMINALLY | FAIRLY |
| LARGELY | HARDLY | ABSOLUTELY |

Table 4.2 - A Sample of Linguistic Hedges

An outstanding feature of linguistic modifiers is their ability in providing means of approximating a fact intended as a premise of a rule. In this way, in the realm of fuzzy logic, a conclusion can be deducted even if the premise is only roughly or incompletely satisfied [BOUC92]. The topics of implication rules and fuzzy expert systems are approached in the next section.

4.4- Fuzzy Expert Systems

4.4.1 - General Discussion

The initial project for developing artificial intelligence systems was quite ambitious, and the objective in the early sixties was to build general problem-solving programs, like the system proposed by Newell and Simon [FEIG63]³⁸. Due to preliminary unsuccessful attempts in this direction, the focus turned to the discovery efficient algorithms capable of dealing with large and complex data structures, but limited to specific subject domains [FOGE95]. So, the term *expert system* was coined, and its functioning relied on information that had been previously retrieved from human experts and stored in the form of a program in a computer, usually employing the language LISP (1962) and later PROLOG (1972).

An expert system consists primarily of a knowledge base and an inference engine, which searches the former in order to check if there are any facts that match the queries posed to the system.

Example of representation of *facts* and a rule in pseudo-Prolog code:

- `high(Price, Price ≥ 5);` \Rightarrow fact
- `very_sensitive(S, S ≥ 2.5);` \Rightarrow fact
- `if {(Price ≥ 5) and (S ≥ 2.5)}` \Rightarrow rule
`then attractiveness(Price, unattractive);`

When a query such as ? - `attractiveness(Price, X)`³⁹, the inference engine explores the structure of facts and rules to determine the category of X.

If the elements in the knowledge are *conjectures*, rather than facts, and diverse truth values (or degrees of confidence) are ascribed to those elements by specialists, than a collection of production rules of the type `< IF premise THEN conclusion >` can be created and assessed in order to answer a variety of queries posed to the system, even though the conjectures are matched only partially.

³⁸ "GPS: A Program that Simulates Human Thought"

³⁹ X is a linguistic variable that qualifies the numeric variable "Price" in terms of "attractiveness."

The first program to use this schema was DENDRAL⁴⁰, started in the mid-sixties and improved along the subsequent years. The well-known medical program MYCIN⁴¹ used several ideas successfully introduced by the DENDRAL project. Another outstanding expert system was PROSPECT, which was employed as a geological exploration aid and had more than one thousand rules [FOGE95].

Despite their relative success and accomplishments, expert systems have been sometimes criticized in their ability to represent human intelligence. One point often mentioned regards the difficulty of capturing the behavior of complex cognitive systems with an assortment of facts and rules. As a reply to those critics, it should be said that perhaps too much is being expected from expert systems in terms “intelligence” as only mirroring what is generally accepted and understood as the human course of action. Unquestionably, a human decision maker, while much more versatile in the mental manipulation and association of facts, hypothesis and conclusions, usually falls behind the performance of an expert system when the decision tree has too many branches and levels. In those circumstances, the computer on which the expert system is running is then playing the role of a *facts-and-rules-cruncher*, in substitution of the customary denomination of *number-cruncher*, and this task is significant in the treatment and solution of an a great number of problems in assorted areas of interest.

A question that has been recently occupying the thoughts of researchers in the field of artificial intelligence is the counterpoise of *expert knowledge* and *general intelligence*. Hofstadter [HOFS95] comments about what he calls the *expert systems trap*, “... the idea that the key to all intelligence is just knowledge, knowledge, and ever more knowledge.” He admits that “some domain knowledge is necessary to get off the ground” but understands intelligence as having “a powerful, general and abstract knowledge-independent core”.⁴²

⁴⁰ DENDRAL is a chemistry program that provided information about the molecular structure of unknown compounds, based on their mass spectral and nuclear magnetic response data.

⁴¹ MYCIN diagnosed bacterial infection in hospital patients.

⁴² Hofstadter testifies that he has known “many people who were fountains of knowledge yet seemed to lack all insight”.

The capacity to adapt itself to dynamic environments and learn from experience are also important attributes in proficient expert systems⁴³. Furthermore, in most instances, the achievements of expert systems can be greatly enhanced by associating them with other up-to-date *soft computing* techniques, such as neural networks, genetic algorithms and fuzzy logical inference, yielding what is commonly called a *hybrid system* [HALL94]. Some fundamental topics regarding fuzzy logic and inference are presented in the next section.

4.4.2 - Fuzzy Logic and Fuzzy Inference

In the same manner that fuzzy sets are an extension of ordinary sets, Fuzzy Logic is, in most aspects, a generalization of classical logic, and many of the basic concepts of the latter are also valid for the former.

Consider that X and Y represent, respectively, the universes of discourse of the two variables *price* (x) and *quality* (y) of a product being sold in the market by a Firm. Suppose that x is qualified by the predicates $\{A_j\}$, $j=1, 2, \dots, m$ and y by $\{B_j\}$, $j=1, 2, \dots, n$.

A statement involving x and a_j for example, could be: x is a_1 . In traditional binary or Boolean logic, a statement can only be either true or false. Under this assumption, the *truth-value* $t_S(x)$ of any statement S regarding x equals 1 if S is true, or 0, if S is false.

A composite assertion I_1 comprising at the same time the variables x and y of the type

$$I_1: \underbrace{\text{IF } x \text{ is } A_j}_{\text{statement P}} \quad \underbrace{\text{THEN } y \text{ is } B_j}_{\text{statement Q}}$$

⁴³ An example of a dim-witted biological expert system is given by the French naturalist J. H. Fabré, cited in Gould and Gould [GOUL85] and [FOGE95]. When the female wasp *Sphex flavipennis* must lay its eggs, it builds a burrow and paralyzes a hunted cricket with venom. The cricket is intended to serve as food for the grubs after the eggs hatch. It is then taken to the entrance of the burrow and before dragging it inside, the wasp enters the lair for inspection. If the cricket is moved from the place where the wasp put it while it was inside the burrow, the wasp repeats the same procedure of placing the cricket in a particular position and entering the hole for a new inspection, no matter how many times this is done. The wasp uses a rule based system that is incapable of learning with the experience, "astounding us with its extraordinary intelligence, and, in the next moment, surprises us with its stupidity, when confronted with a simple fact that departs from its ordinary practice."

is denominated a *production, inference or implication rule*. The first part of I_1 [statement P] is designated *antecedent* or *premise*; the second segment [statement Q] is the *consequent* or *conclusion*. I_1 is an example of classical *modus ponens*, the most simple inference rule [GRIF87].

Table 4.3 presents a list of the Boolean truth-values regarding the statements P, Q, the disjunction $(P \vee Q)$, the conjunction $(P \wedge Q)$, and the implication rule $P \rightarrow Q$.

| Truth-values (t) | | | | |
|----------------------|-------|------------------|----------------|-----------------------|
| t_P | t_Q | $t_{P \wedge Q}$ | $t_{P \vee Q}$ | $t_{P \rightarrow Q}$ |
| 0 | 0 | 0 | 0 | 1 |
| 0 | 1 | 0 | 1 | 1 |
| 1 | 0 | 0 | 1 | 0 |
| 1 | 1 | 1 | 1 | 1 |

Table 4.3 - A Boolean truth-table for the statements P, Q.

Restraining the possible truth-values to only *zero* and *one* conveys an immediate repercussion, that is the **Law of the Excluded Middle**. The operator **not** (denoted by the symbol \neg), when applied to a Boolean proposition, reverses its truth-value from one to zero and vice-versa. This entails no possibility of a statement S to be simultaneously true and false. Nevertheless, philosophical considerations about logical paradoxes caused by the Law of the Excluded Middle were one of the factors that led the polish logician Jan Lukasiewicz to the development of a three- valued logic, with truth values of $\{0, \frac{1}{2}, 1\}$, which was subsequently extended to all values in $[0,1]$.

If the restriction of confining the potential truth-values to $\{0, 1\}$ is abandoned and $t_S(x)$ is allowed to assume any point in the interval $[0, 1]$, then the *modus ponens* inference rule can be generalized, ensuing fuzzy implication rules. This generalization signifies that a *perfect* match between an observed fact and the rule's premise is not anymore required to its firing. Furthermore, the predicates A_j, B_j may be fuzzy variables or fuzzy numbers.

In this way, an *approximate* likeness between a fact and a rule's antecedent is a sufficient condition for the rule's activation. But this activation is not restricted to *all or nothing* either, and the conclusion's truth-value will depend on the antecedent's degree of confidence. That schema elicits a number of fuzzy implication rules. Zadeh's original proposal for implication defines [SMIT87]

IF P then Q else O⁴⁴

as either (P and Q) or (\neg P and O)⁴⁵.

The grade of membership of an element in a fuzzy set may correspond to the truth value ($t_{P \rightarrow Q}$) of a proposition $P \rightarrow Q$. $t_{P \rightarrow Q}$, besides depending on t_P and t_Q , is also a function of the particular implication rule taken into account. A fuzzy implication can be interpreted as describing a relation between two fuzzy sets [MUNA94]. Some important fuzzy implication rules and their correspondent truth-values are displayed below.

- $t_{P \rightarrow Q} = \max(\min(t_P, t_Q), 1 - t_P)$ (Max-min implication)
- $t_{P \rightarrow Q} = \max(1 - t_P, t_Q)$ (Arithmetic implication)
- $t_{P \rightarrow Q} = \begin{cases} \frac{t_Q}{t_P} & \text{if } t_P \geq t_Q \\ 1 & \text{otherwise} \end{cases}$ (Goguen's implication)
- $t_{P \rightarrow Q} = \min(1 - t_P + t_Q, 1)$ (Lukasiewicz' implication)
- $t_{P \rightarrow Q} = \min(t_P, t_Q)$ (Mamdani's implication)
- $t_{P \rightarrow Q} = \begin{cases} 1 & \text{if } t_Q \geq t_P \\ t_Q & \text{otherwise} \end{cases}$ (Gödelian implication)

⁴⁴ A synthetic expression for "If P is true, then Q is true, otherwise R is true".

⁴⁵ If O is not specified, it is interpreted as the universe, and (\neg P and O) becomes simply \neg P.

$$\bullet \quad t_{P \rightarrow Q} = \begin{cases} 1 & \text{if } t_Q \geq t_P \\ 0 & \text{otherwise} \end{cases} \quad (\text{Rescher's implication})$$

$$\bullet \quad t_{P \rightarrow Q} = 1 - t_P + t_P \times t_Q \quad (\text{Reichenbach's implication})$$

The input of a *modus ponens* inference rule may be composed of multiple antecedents, which are linked to each other by a connective, usually AND or OR.

The steps to perform a fuzzy inference from P to Q are:

1. Define the implication rule, e.g. IF P (x is A_1) THEN Q (y is B_1), $x \in X$, $y \in Y$.
2. Determine the truth-value t_P of P, which corresponds to the grade of membership of x in the fuzzy set A_1 : $\mu_{A_1}(x)$ (fuzzification of x);
3. Induce the degree of membership of y in B_j : $\mu_{B_j}(y)$, applying the pertinent fuzzy relation between A and B , which is defined by the Cartesian product⁴⁶ $A_j \times B_j$.
4. Perform a defuzzification of $\mu_{B_j}(y)$, obtaining y^∇ .

4.4.2 - Defuzzification Methods

The fuzzification procedure consists in determining the membership grade of an element in a fuzzy set applying the respective pre-defined function. If multiple inputs are involved, chained disjunction or conjunction operations are performed, depending on whether the connectives are AND or OR. On the other hand, the reverse process, that is, given the membership grades of the variable in the applicable fuzzy subsets, find the numeric value y^∇ , can be accomplished by various methods. The problem is always to select an algorithm that can perform in the most appropriate way, the task of converting membership degrees in a single output on the real line.

⁴⁶ $A_j \times B_j = \{ \min [\mu_{A_j}(x), \mu_{B_j}(y)] / (x, y) \mid x \in X, y \in Y \}$.

There is not such a thing as *the best method*. The ~~advantages of preferring one~~ over another regarding the quality of the results depend heavily on the characteristics of the system that is being modeled.

Among the methods that are mostly often found in the literature and employed in the actual fuzzy systems [YAGE93b] are⁴⁷:

- The *Center of Area Method* - COA (also called *center of gravity* or *centroid*):

$$y^{\nabla} = \frac{\int_{-\infty}^{+\infty} y \mu_B(y) dy}{\int_{-\infty}^{+\infty} \mu_B(y) dy} \quad \text{Eq. 4.22}$$

- The *Mean of Maxima Method* - MOM:

$$y^{\nabla} = \frac{1}{m} \sum_{B_j} \max \mu_{B_j}(y), \quad \text{Eq. 4.23}$$

where m is the cardinality of $B_j \mid \exists \mu_{B_j}(y) \neq 0$.

When the COA defuzzification method is implemented, it is quite often the case that multiple rules are fired, so that the conclusion is described by two or more fuzzy subsets that overlap. In these circumstances the *centroid* may be calculated either considering the area regarding each subset as independently contributing to the computation, or avoiding double-counting the intersection, thus eliminating the overlapped portion of the area. In the program built to simulate the model of the market share game in Chapter 6, the former alternative was selected [KOSK93].

The defuzzification process can be significantly simplified if the *singleton* method is adopted. Instead of the center of gravity of an *area*, like in the COA method, the centroid is determined taking into account the rules' strengths and the corresponding points on the horizontal axis. Hence, a singleton is a single vertical line that stands for the membership function [VIOT93]. Generally, the point on the horizontal axis that

⁴⁷ As a matter of fact, defuzzification methods are constantly being refined and enriched by new techniques. For a recent and illustrative survey, see op. cit. [YAGE94], p. 313-355.

corresponds to the membership grade is not unique, so the middle points of each segment are chosen as the x value of the particular activated fuzzy set to which the segment belongs.

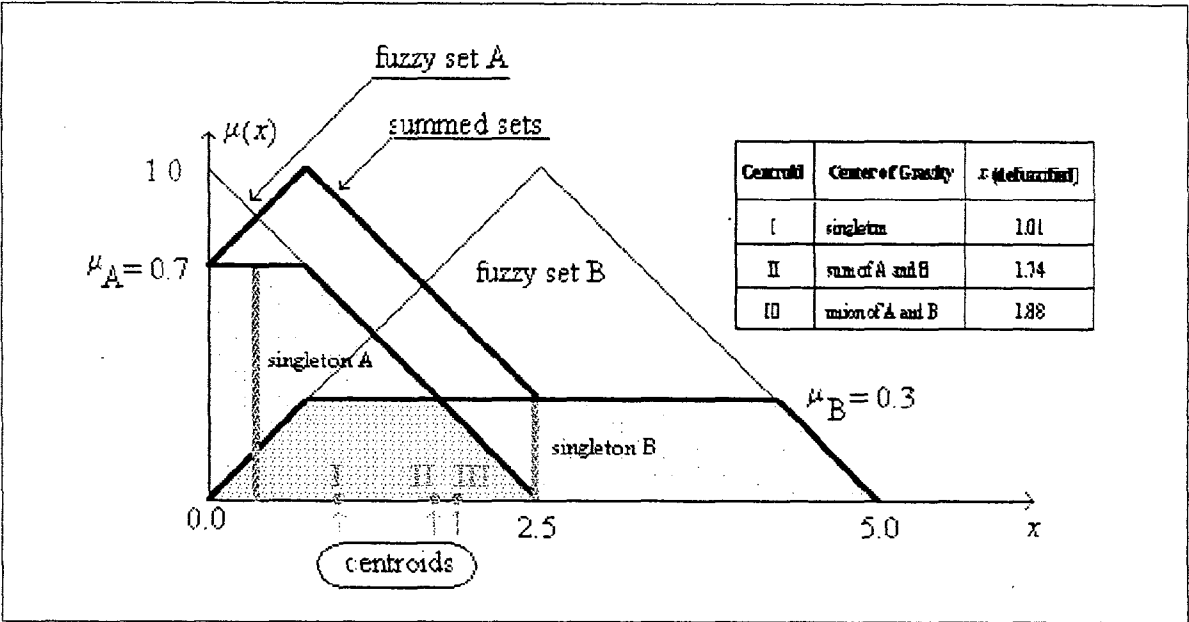


Figure 4.6 - Example of Three Criteria for determining the Centroid using the Center of Gravity Method (COA)

Figure 4.6 illustrates the defuzzification procedure using the center of gravity method, with three alternatives regarding the computation of the centroid. There is not a big difference between criteria II and III. The singleton yields quite distinct results if compared to the former values of x , which, nonetheless, might also be appropriate depending on the system being modeled. For the sake of objectiveness and minimization of computational burden, the defuzzification method employing singletons was selected to promote the simulations of the Fuzzy Iterated Prisoner's Dilemma game—FIPD—, to be presented next, in Chapter 5.

Chapter 5

A Fuzzy Approach to the Prisoner's Dilemma

5.1 - Introduction

In this part of the dissertation, a model of the *Fuzzy Iterated Prisoner's Dilemma* is presented and developed. The objective that is aimed with this treatment, allowing the players to depart from the traditional dichotomic moves is the achievement of a deeper understanding of more realistic PD-like situations. The possibility of acting within a continuous range of choices is much more similar to those commonly found whenever a conflict of interest arise than the usual restriction to only two mutually exclusive selections..

The traditional IPD assumes a binary choice for the players, either *cooperation*(C) or *defection*(D). The implementation of the strategies generates sequences of these two kinds of moves. The resulting payoffs are numerical values specified in the cells of a payoff matrix according to a particular pair of decisions made by the participants. Some researchers have considered the departure from the binary choice. For example, Hardin (1982) takes gradual cooperation into account mainly in the pursuit of divisible public goods, and Harrald and Fogel [HARR95] employ a planar approximation of Axelrod's payoff matrix as a basis for an evolutionary simulation of the IPD, but use neural networks to represent the players' strategies.

Much of the present interest in the IPD derives from Axelrod's computer tournaments [AXEL84], where diversified strategies were confronted with each other and the winner, TIT-FOR-TAT (TFT), proved robust in several other simulations of the IPD. Axelrod, in a subsequent work [AXEL87], used a genetic algorithm to determine if TFT could evolve from a random set of strategies.

The *Fuzzy Iterated Prisoner's Dilemma - FIPD* diverges from the usual classic *C* or *D* binary form, and thereby the moves selected by the players are mapped in a scale ranging from 0 to 1, these extreme points corresponding, respectively, to full defection and full cooperation. Since the inputs of the game are no longer discrete, a payoff function consisting of two intersecting planes is used as a substitute of the payoff matrix.

Allowing the moves to vary continuously in an interval brings a wider range of possibilities to the IPD. But the main subject to be explored in this Chapter is the representation of every player by a specific *fuzzy expert system* (FES). The participants will perceive and implement the actions pertaining to the game only qualitatively, according to individual rules. Then, the expert systems transform the qualitative strategies into numerical values, which will be the inputs of the payoff function.

The underlying idea resembles the way a person drives a car. Depending on the vehicle trajectory, the driver assesses the situation and reacts steering to the right or left, with different emphasis. In this way, though the driver succeeds in maintaining the car under control, it does not know the exact angle and other numerical characteristics of its action conveyed to the wheels by the mechanical system of the car. Here, likewise, the players are represented by their respective FES, which will sense, measure and implement the actions that belong to the game, where the moves, though gradual, are still labeled as cooperation (C) or defection (D).

However, considering the subjective interpretation of these two concepts (C and D), they have been replaced by the game payoffs, taken under three different fashions, as the players' sole inputs to generate their decisions. Three FES's are assigned to every participant, each one having as input a distinct deterministic factor based on the payoffs that take place during the game, and as the output either COOPERATE or DEFECT, but with variable strength.

The universe of discourse of each input factor is divided into three fuzzy sets, and this schema yields $2^3 = 8$ different strategies for any particular FES. The total number of strategies for all three FES is then $8^3 = 512$, which corresponds to the population of *fuzzy strategists* or *fuzzy players*. In addition, both the "Pavlov" [BEAR93] and the TFT

strategists are present in the tournament. The purpose was checking how well these well-known successful strategies fare when confronted to the standard players considered in this paper, that is, those whose actions are guided by the respective FES.

5.2 - A Payoff Function

The payoff matrix of the classical IPD game is shown in Table 5.1 [AXEL84].

| Row Player's Decisions (Player 1) | Column Player's Decisions (Player 2) | |
|-----------------------------------|--------------------------------------|---------|
| | Cooperate | Defect |
| Cooperate | R R | S T |
| Defect | T S | P P |

Table 5.1 - Usual Payoff Matrix of the PD game

The pair of values in the cells represents the payoffs of the row and the column players, respectively. The meaning of the letters standing for the payoffs are:

- R: Reward for mutual cooperation
- S: The “sucker’s” payoff
- T: Temptation to defect
- P: Punishment for mutual defection

To represent a IPD, the values T, R, P, and S in Table 5.1 must obey the Equations 5.1 and 5.2. In the classic IPD game, a commonly adopted set of payoffs is T = 5, R = 3, P = 1, and S = 0.

$$T > R > P > S$$

Eq. 5.1

$$\frac{T + S}{2} < R$$

Eq. 5.2

But in the FIPD, instead of punctual choices, COOPERATION and DEFECTION are considered as two non-overlapping fuzzy sets, and a particular move, or action (a) is defined by the *degree of membership* $\mu_C(a)$ or $\mu_D(a)$ in one of these sets.

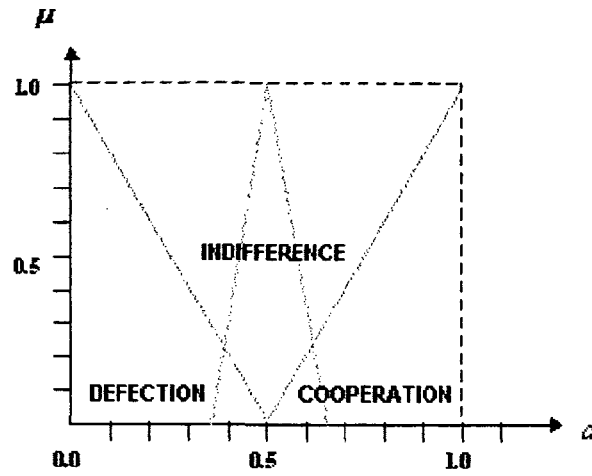


Figure 5.1 - Fuzzy sets describing possible actions in the FIPD

The universe of discourse of the actions has been divided into only two mutually exclusive triangular fuzzy sets C and D, but other shapes could be adopted. For instance, a fuzzy set expressing INDIFFERENCE, centered on $\alpha=0.5$ could have been added, as shown in Figure 5.1¹. The degree of membership of an action in either fuzzy set COOPERATE or DEFECT is given by Equations 5.3(a) and 5.3(b), respectively.

$$\mu_C(a) = \begin{cases} 2a - 1 & ; \text{when } a \geq 0.5 \\ 0 & \text{otherwise} \end{cases} \quad \text{Eq. 5.3(a)}$$

$$\mu_D(a) = \begin{cases} -2a + 1 & ; \text{when } a \leq 0.5 \\ 0 & \text{otherwise} \end{cases} \quad \text{Eq. 5.3(b)}$$

Now, once the players can make moves anywhere in the interval $[0,1]$ (DEFECT, COOPERATE), the typical payoff matrix shall be replaced by a linear function of two independent variables that stand for each player's action. The output, as shown in Figure 5.2, consists of the resulting payoff for player 1, depicted by two intercepting planes.

¹ Using two fuzzy sets to characterize the actions generate 8 rules for each decision factor (See section III). Since there are three decision factors, the total number of strategies will be 512 ($2^3 \times 2^3 \times 2^3$). If a third fuzzy set were added, this number would increase to 19683 ($3^3 \times 3^3 \times 3^3$), each one corresponding to a different player in the game.

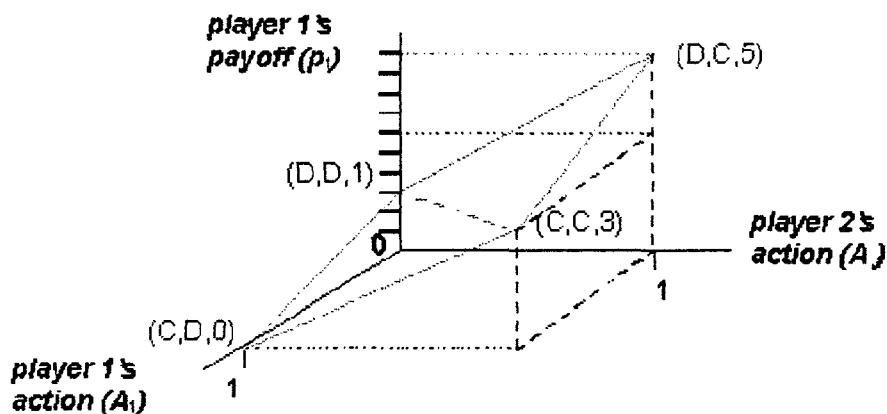


Figure 5.2 - The FIPD's payoff function

The payoffs corresponding to the combinations of the extreme actions C and D ($\mu_C(a)=1$ and $\mu_D(a)=1$, respectively) were arbitrarily chosen as $T = 5$, $R = 3$, $P = 1$, and $S = 0$, which correspond to commonly employed values. The players' gains are given by Equations 5.4(a) and 5.4(b).

a) Player 1's payoff (p_1):

$$p_1 = \begin{cases} 1 - 2a_1 + 4a_2; & \text{when } a_1 < a_2 \\ 1 - a_1 + 3a_2; & \text{when } a_1 \geq a_2 \end{cases} \quad \text{Eq. 5.4(a)}$$

b) Player 2's payoff (p_2):

$$p_2 = \begin{cases} 1 - 2a_2 + 4a_1; & \text{when } a_2 < a_1 \\ 1 - a_2 + 3a_1; & \text{when } a_2 \geq a_1 \end{cases} \quad \text{Eq. 5.4(b)}$$

5.3 - Fuzzy Decision Rules

A quite common criteria to model the players' strategies, also used here, is to base them on the sequence of the opponent's last moves. Nevertheless, two other variables that seem to be related to the player's choices are added to the present model. The variables are represented by f_1 , f_2 and f_3 , henceforth called decision factors, and defined as:

f_1 : Relation between the accumulated wealth of the player and his opponent's;

f_2 : Last iterations between the parties;

f_3 : Relation between the overall trends regarding the acquisition of wealth, expressed by the player's average gain per iteration divided by the global payoff mean of the whole group.

In the sequence, each factor is detailed, including how they were modeled into fuzzy sets and fuzzy decision rules.

5.3.1 - Relation between accumulated wealths: f_1

One of the criteria that might affect a player's decision is the relation between his own wealth and that possessed by the opponent. The idea that underlies the adoption of the factor f_1 comes from Maynard Smith's *Hawk-Dove Game* [SMIT82]. There, an element called *Resource Holding Power* - RHP, which stands for a contender's measure of size, strength, weapons, or other kinds of power symbols, plays a significant part on the resolution of a dispute.

The RHP component influences the formulation and efficacy of strategies. If a player knows, for instance, that its opponent has a much higher wealth, it is intuitively aware that this fact can act upon its partner's decisions, perhaps making it more daring because it is confident of its relative ascendance. On the other hand, a favorable relation can also be used by the player as a hint that it is performing fairly well with its strategy if compared to a current competitor. Also, a player with a high f_1 can implement strategies

with less concerns about temporary losses. Equation 5.5 considers the relation between a player whose wealth is w_0 and an opponent with w_1 .

$$f_l = \frac{w_0}{w_0 + w_1}, f_l \in [0, 1], \forall w_0, w_1 \quad \text{Eq. 5.5}$$

To qualitatively characterize f_l , three fuzzy sets have been designed: MUCH LOWER (LW), SIMILAR (SL), and MUCH GREATER (GT), which are the same for all players (Figure 5.3).

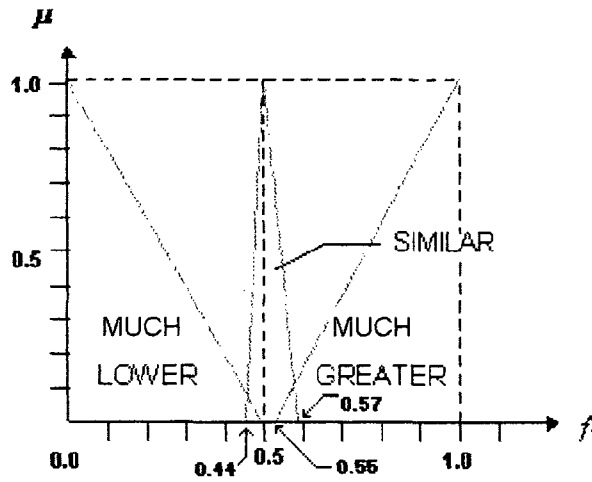


Figure 5.3 - Fuzzy Sets for the qualitative description of f_l

In the fuzzy set SIMILAR, $\mu_{\text{EQ}}(f_l=0.5)=1$, and the left and rightmost points ($\mu_{\text{EQ}}(f_l')=\mu_{\text{EQ}}(f_l'')=0$) were arbitrarily chosen as $f_l'=0.44$ and $f_l''=0.57$, trying to mirror an asymmetry usually found in human reasoning. This means that a player would restrain its concept of *equivalence* within an interval where its wealth is 20% lower and 30% greater than the opponent's (*i.e.* points $w_0=0.8w_1$ and $w_0=1.3w_1$). For a similar reason the third fuzzy set greater starts at $f_l=0.55$, which results from $w_0=1.2w_1$, approximately. The degree of membership of f_l in each of those fuzzy sets is given by Equations 5.6(a) - 5.6(c).

$$\mu_{\text{LW}}(f_l) = \begin{cases} -2f_l + 1 & \text{for } 0 \leq f_l \leq 0.50 \\ 0 & \text{otherwise} \end{cases} \quad \text{Eq. 5.6(a)}$$

$$\mu_{SL}(f_1) = \begin{cases} \frac{30}{3} f_1 - \frac{22}{3} & \text{for } 0.44 \leq f_1 \leq 0.50 \\ -\frac{100}{7} f_1 - \frac{57}{7} & \text{for } 0.50 \leq f_1 \leq 0.57 \\ 0 & \text{otherwise} \end{cases}$$

Eq. 5.6(b)

$$\mu_{GT}(f_1) = \begin{cases} \frac{20}{9} f_1 - \frac{11}{9} & \text{for } 0.55 \leq f_1 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Eq. 5.6(c)

The factor f_1 is used as an antecedent in a set of fuzzy production rules, whose consequents are either gradual COOPERATION or DEFECTION. The same mechanism is employed for the other two decision factors f_2 and f_3 . The conclusions yielded separately by each expert system relative to f_1 , f_2 and f_3 are partial, and the final decision shall be made conjointly, as will be explained in the sequence. Regarding f_1 alone, the possible combinations of rules are listed in Table 5.2. For each strategy $S^i_{f_1}$ the possible consequents (C or D) are shown in the shaded cells.

| Strategies ² | | | | | | | | |
|--------------------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| Wealth Relation f_1 | $S^1_{f_1}$ | $S^2_{f_1}$ | $S^3_{f_1}$ | $S^4_{f_1}$ | $S^5_{f_1}$ | $S^6_{f_1}$ | $S^7_{f_1}$ | $S^8_{f_1}$ |
| MUCH LOWER | C | C | C | C | D | D | D | D |
| SIMILAR | C | C | D | D | D | D | C | C |
| MUCH GREATER | C | D | C | D | D | C | D | C |

Table 5.2 - Fuzzy Production Rules involving the wealth relation f_1 and an action a_i

² $S^i_{f_1}$: Strategy i in relation to the factor f_1

5.3.2. Last Iterations Between The Parties: f_2

The factor f_2 refers to the influence of the recent history of a contender's moves. It consists of a relation between a weighted average of the payoffs obtained in the last three iterations and the maximum achievable payoff, i.e., 5. When it is the case that two specific players have not met three times yet, the default values specified by Equations 5.7(a)-5.7(c) are used.

No previous mutual iterations:

$$f_2 = 0.4 \quad \text{Eq. 5.7(a)}$$

One previous mutual iteration:

$$f_2 = \frac{p_i^{t-1}}{5} \quad \text{Eq. 5.7(b)}$$

- Two previous mutual iterations:

$$f_2 = 0.4 \times \frac{p_i^{t-2}}{5} + 0.6 \times \frac{p_i^{t-1}}{5} \quad \text{Eq. 5.7(c)}$$

- Three previous mutual iterations:

$$f_2 = 0.1 \times \frac{p_i^{t-3}}{5} + 0.3 \times \frac{p_i^{t-2}}{5} + 0.6 \times \frac{p_i^{t-1}}{5} \quad \text{Eq. 5.7(d)}$$

The terms $\frac{p_i^{t-v}}{5}$ ($v = 1, 2, 3$) stand for the relations between the gain that a player obtained and the maximum payoff admissible in previous iterations.

The factor f_2 belongs to $[0, 1]$ and accounts for the relative gain achieved in an iteration. It is described by the attributes POOR, FAIR, and HIGH, which correspond to the respective fuzzy sets shown in Figure 5.4 and membership grades given by Equations 5.8(a) to 5.8(c). The points $f_2 = 0.2$ and 0.6 were determined with a fixed payoff $p_i=1$ and

$p_i=3$, respectively, which conform to the pairs of dichotomic moves CC and DD. As to $f_2 = 0.4$, it was calculated with $p_i=2$, originated from $\mu_C(a)=\mu_D(a)=0$ for both players³.

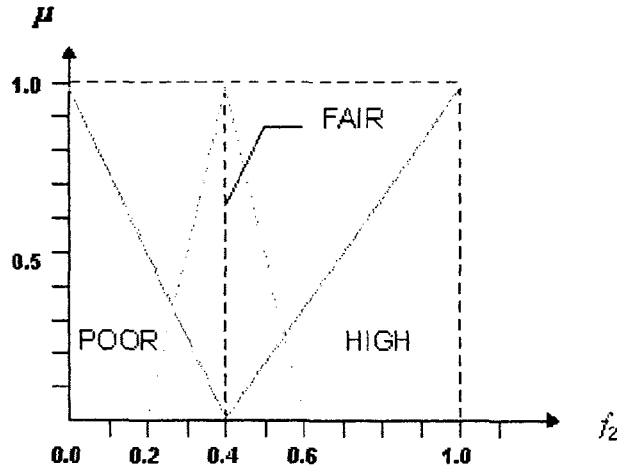


Figure 5.4 - Fuzzy Sets for the qualitative description of f_2

The fuzzy expert systems which employ f_2 follow the same pattern adopted for f_1 , only substituting POOR, FAIR and HIGH for LOWER, SIMILAR and GREATER in Table 5.2.

$$\mu_{\text{POOR}}(f_2) = \begin{cases} -2.5f_2 + 1 & \text{for } 0 \leq f_2 \leq 0.40 \\ 0 & \text{otherwise} \end{cases} \quad \text{Eq. 5.8(a)}$$

$$\mu_{\text{FAIR}}(f_2) = \begin{cases} 5f_2 - 1 & \text{for } 0.20 \leq f_2 \leq 0.40 \\ -5f_2 + 3 & \text{for } 0.40 \leq f_2 \leq 0.60 \\ 0 & \text{otherwise} \end{cases} \quad \text{Eq. 5.8(b)}$$

$$\mu_{\text{HIGH}}(f_2) = \begin{cases} \frac{5}{3}f_2 - \frac{2}{3} & \text{for } 0.40 \leq f_2 \leq 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{Eq. 5.8(c)}$$

³ From this point on, with any combination of actions a player will receive a greater payoff than its opponent. For this reason, $f_2 = 0.4$ was chosen as the limit between the POOR and the HIGH fuzzy sets. As to fuzzy set FAIR its limiting points have been determined under two assumptions: First, a f_2 below 0.2 means that the player is being exploited by the other party and, thus, cannot see the outcome as "fair". Second, in an opposite sense, if it obtains more than 3 points ($f_2 > 0.6$), this reveals an advantageous situation over the opponent, implying in gains that might not be considered "fair".

5.3.3. Relation Between The Overall Trends Of Wealth: f_3

The third factor f_3 is a linear relation between a player's (\bar{w}_r^k) and the population's (\bar{W}^n) average payoffs per iteration. This relation intends to portray the current trend in the accumulation of gains.

$$\bar{w}_r^k = \frac{\sum_{j=1}^k w_r^j}{k}$$

Eq. 5.9(a)

$$\bar{W}^n = \frac{\sum_{j=1}^n \sum_i w_i^j}{2n}$$

Eq. 5.9(b)

$$f_3 = \frac{\bar{w}_r^k}{\bar{w}_r^k + \bar{W}^n}$$

Eq. 5.9(c)

where:

- w_r^j is the wealth of the player r at the j^{th} iteration,
- \bar{w}_r^k is the average wealth per iteration that the player has received until iteration k
- k is the number of iterations already performed by the player r ,
- \bar{W}^n is the average wealth per iteration that the whole population has received after n iterations,
- n is the total number of iterations until the present instant,
- $i \in P, P: \{\text{players who have already played}\}$

The fuzzy sets designed to represent the qualitative concepts associated to f_3 are similar to those shown in Figure 5.3, and eight fuzzy production can be generated in the same manner as depicted in Table 5.2.

$$\mu_{\text{LW}}(f_3) = \begin{cases} -2f_3 + 1 & \text{for } 0 \leq f_3 \leq 0.50 \\ 0 & \text{otherwise} \end{cases} \quad \text{Eq. 5.10(a)}$$

$$\mu_{SL}(f_3) = \begin{cases} \frac{50}{3} f_3 - \frac{22}{3} & \text{for } 0.44 \leq f_3 \leq 0.50 \\ -\frac{100}{7} f_3 + \frac{57}{7} & \text{for } 0.50 \leq f_3 \leq 0.57 \\ 0 & \text{otherwise} \end{cases} \quad \text{Eq. 5.10(b)}$$

$$\mu_{GT}(f_3) = \begin{cases} \frac{20}{9} f_3 - \frac{11}{9} & \text{for } 0.55 \leq f_3 \leq 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{Eq. 5.10(c)}$$

5.4 - Determination of a Player's Action in a Move

The evaluation of f_1 , f_2 , and f_3 yields three separate partial conclusions about the intensity of cooperation/defection to be adopted. To accomplish the aggregation of the three limited results, the present model employs a procedure that takes the maximum of the outcomes derived from the rules related to each factor. The general steps to be followed by a player r ($r = 1$ to m) are valid for every factor $f_{q,r}^j$ (q corresponds to the type of the factor ($q = 1, 2$ or 3) and j to the iteration ($j = 1$ to n)).

- i. *Determination of $f_{q,r}^j$;*
- ii. *Fuzzification of $f_{q,r}^j$;*
- iii. *Activation of the fuzzy rule(s) matched by the fact (according to the player's strategy $S_{q,r}$);*
- iv. *Consolidation of the partial conclusions yielding the final conclusions (μ_C^r, μ_D^r);*

Every player is characterized by a set of 9 fuzzy rules (three for each $S_{q,r}$, $q=1, 2, 3$). The inference process will fire the rules matched by the fuzzified decision factors. Each factor can originate either one or any pair of the possible partial conclusions C or D, with its associated strength (membership grade μ). The integration of the partial conclusions is made through the fuzzy union, using the join operator \vee (fuzzy maximum), for the same class of conclusion. Hence, the final result to be derived in terms of cooperation and defection will be a sole pair of conclusions, taken as $\max(\mu_C^{fq})$ and $\max(\mu_D^{fq})$, $q = 1, 2$ and 3 .

v. Defuzzification of the final conclusions to determine the final player's action A_t^j .

The final conclusion will consist of a pair of membership grades in either one or both fuzzy sets COOPERATE and DEFECT. However, in order to find the payoffs in an iteration, the players' actions cannot be fuzzy. A defuzzification procedure using the *singleton method*⁴ translates the qualitative conclusions into a crisp value for each player.

5.5 - Example of an Iteration of the FIPD

5.5.1 - Identification of the Players

The diagram depicted in Table 5.3 shows how a player is distinguished by the set of strategies it uses. Each player has 9 rules defining what it will do according to the value of each decision factor. Let us assume that one of the players randomly chosen is P_{764} . The subscript 764 means that it uses the strategy $S_{f_1}^7$ with respect to f_1 , $S_{f_2}^6$ for f_2 , and $S_{f_3}^4$ for f_3 , as indicated by the shaded cells.

| f_1 | S_1 | S_2 | S_3 | S_4 | S_5 | S_6 | S_7 | S_8 | f_2 | S_1 | S_2 | S_3 | S_4 | S_5 | S_6 | S_7 | S_8 | f_3 | S_1 | S_2 | S_3 | S_4 | S_5 | S_6 | S_7 | S_8 |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| LW | C | C | C | C | D | D | D | D | POOR | C | C | C | C | D | D | D | D | LW | C | C | C | C | D | D | D | D |
| SL | C | C | D | D | D | D | C | C | FAIR | C | C | D | D | D | D | C | C | SL | C | C | D | D | D | D | C | C |
| GT | C | D | C | D | D | C | D | C | HIGH | C | D | C | D | D | C | D | C | GT | C | D | C | D | D | C | D | C |

Table 5.3 - Strategies used by the 512 different players of the FIPD⁵

5.5.2 - The Iteration Process

Assume that the other player, also randomly selected, was P_{177} . The iteration process is as follows: First, each player determines its decision factors. Every f is qualitatively described by its corresponding fuzzy sets, and the rules that have been

⁴ A singleton is a membership grade value represented by a single vertical line that intercepts the horizontal axis in only one point [VIOT93]. In our case, there will be two singletons, one for each fuzzy conclusion, COOPERATE and DEFECT. The final action is determined by the point where the resultant of the two singletons crosses the horizontal axis. This point is calculated balancing the moments of the two singletons and of the resultant. See Chapter 4, section 4.4.2.

⁵ A player is a strategist with three rules for each of the three decision factors. Thus, the total of players is $8 \times 8 \times 8 = 512$.

activated are added to the knowledge base. After the three factors have been considered, the fuzzy inference takes place using the fuzzy rules previously fired, yielding a player's final action. With both final actions as inputs, the payoffs are calculated using Equations 5.4a and 5.4b. The entire decision process for player P_{764} is described below (P_{177} implements a similar process to reach its action).

5.5.3 - The Decision Process for Player P_{764}

The data assumed for the player P_{764} is:

- $w_{764}^{24} = 36$ (wealth reached after 24 iterations)
- $w_{177}^{30} = 96$ (wealth reached by the current opponent after 30 iterations)
- $\bar{w}_{764}^{24} = 1.5$ (average wealth per iteration after 24 iterations - Eq. 5.9a)
- $\bar{w}^n = 2.5$ (current population's average gain per iteration - Eq. 5.9b)
- Previous Iterations with the player P_{177} :

| Players' actions | Iteration | |
|------------------|-----------|-----|
| | 1 | 2 |
| P_{764} | 0.5 | 0.2 |
| P_{177} | 0.8 | 0.5 |

a) Determination of the partial conclusions based on factors f_1 , f_2 , and f_3

- factor f_1

$$\text{calculation: Eq. 5.5} \Rightarrow f_1 = \frac{36}{36 + 96} = 0.273$$

$$\text{qualification: Eq. 5.6(a)} \Rightarrow \mu_{\text{LOWER}}(0.273) = 0.454$$

rule fired (strategy $S_{f_1}^7$):

$$\text{IF } f_1 \text{ is MUCH LOWER THEN DEFECT} \Rightarrow \text{DEFECT (0.454)}$$

- factor f_2

$$\text{calculation: Eq. 5.7c} \Rightarrow f_2 = \frac{0.4 \times 3.2}{5} + \frac{0.6 \times 2.6}{5} = 0.568$$

$$\text{qualification: Eq. 5.8(b)} \Rightarrow \mu_{\text{FAIR}}(0.568) = 0.160$$

$$\text{Eq. 5.8(c)} \Rightarrow \mu_{\text{HIGH}}(0.568) = 0.280$$

rules fired (strategy $S_{f_2}^6$):

IF f_2 *is* FAIR *THEN* DEFECT \Rightarrow DEFECT (0.160)

IF f_2 *is* HIGH *THEN* COOPERATE \Rightarrow COOPERATE (0.280)

• factor f_3

$$\text{calculation: Eq. 5.9(c)} \Rightarrow f_3 = \frac{1.5}{1.5 + 2.5} = 0.375$$

qualification: Eq. 5.10(a) $\Rightarrow \mu_{\text{LW}}(0.375) = 0.250$

rule fired (strategy $S_{f_3}^4$):

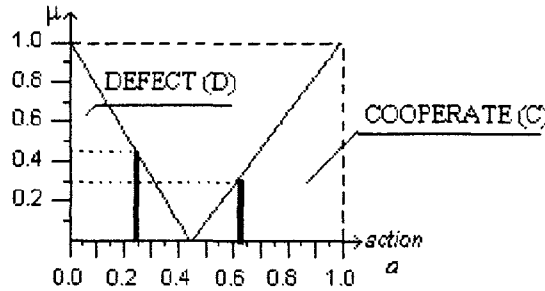
IF f_3 *is* MUCH LOWER *THEN* COOPERATE \Rightarrow COOPERATE (0.250)

b) Combination of the Partial Conclusions (C and D)

$$\mu_C = \max\{\text{COOPERATE (0.280), COOPERATE (0.250)}\} = \text{COOPERATE (0.280)}$$

$$\mu_D = \max\{\text{DEFECT (0.454), DEFECT (0.160)}\} = \text{DEFECT (0.454)}$$

c) Determination of the Final Action of the Player P_{764}



$$\mu_C(a) = 0.280 \therefore a = 0.640 \text{ (Eq. 5.2(b))}$$

$$\mu_D(a) = 0.454 \therefore a = 0.227 \text{ (Eq. 5.2(a))}$$

$$A_{764}^{n+1} = \frac{(1 - \mu_D(a)) \times \mu_D(a) + (1 + \mu_C(a)) \times \mu_C(a)}{2(\mu_C + \mu_D)}$$

$$A_{764}^{n+1} = 0.413$$

5.5.4 - Computation of the payoffs

a) Player P_{764}

$$A_{764}^{n+1} < A_{177}^{n+1}; \quad \text{using Eq. 5.3(a)} \Rightarrow \quad p_{764} = 1 - 2 \times A_{764}^{n+1} + 4 \times A_{177}^{n+1}$$

$$p_{764} = 1 - 2 \times 0.473 + 4 \times 0.532$$

$$p_{764} = 2.182$$

b) Player P_{177}

$$A_{177}^{n+1} > A_{764}^{n+1}; \quad \text{using Eq. 5.4(b)} \Rightarrow \quad p_{177} = 1 - A_{177}^{n+1} + 3 \times A_{764}^{n+1}$$

$$p_{177} = 1 - 0.532 + 3 \times 0.473$$

$$p_{177} = 1.887$$

5.5.5 - Update of the Population's and Players' Parameters

- accumulated wealth (P_{764}): $w_{764}^{25} = (36 + 2.182) = 38.182$
- accumulated wealth (P_{177}): $w_{177}^{31} = (96 + 1.887) = 97.887$
- average payoff per iteration (P_{764}): $\bar{w}_{764}^{25} = 38.182 \div 25 = 1.527$ (Eq. 5.9(a))
- average payoff per iteration (P_{177}): $\bar{w}_{177}^{31} = 97.887 \div 31 = 3.157$ (Eq. 5.9(a))
- population's average payoff per iteration: $\bar{w}^n = 2.5$ (Eq. 5.9(b))

5.6 - Simulations

The computational tournaments were implemented using a custom C++ program, specifically written for the implementation of this model of the FIPD. To run the simulations, the 512 *fuzzy* players were divided in groups of 16 participants each. Along the four different phases of the tournament, every group included also the three non-fuzzy strategists, that is, the traditional TFT, the generalized TFT⁶ (TFT-g) and *Pavlov*.

In the first phase there were 32 groups, each with 19 contenders. After running 30000 iterations per group, the competitors were ranked by their average payoff. For the second round, the eight best fuzzy players from each group were then selected, and eleven new groups were formed from the original 32, by mixing the participants by chance.

The third phase involved five groups, again picked from the foregoing best performers. The 16 final best fuzzy strategists, along with TFT, TFT-g and Pavlov played the last dispute. The total number of iterations per group was kept constant in 30000 in every phase, and there no player performed less than 3000 iterations. This means that, on the average, everyone encountered the same adversary in about 166 occasions.

Tables 5.4 to 5.7 display the results obtained in the four phases of the simulations, always referring to groups of 19 players each (16 fuzzy and the three non-fuzzy).

⁶ The Generalized TFT strategist starts cooperating and, in the next iterations, repeats the previous punctual action of the opponent. Another possibility could be to consider the opponent's action as an α -cut in the fuzzy action (C or D) and use a defuzzification procedure to find its decision.

| Group | Winner's Number | Avg. Payoff | Group's Avg. Payoff | Smallest payoff |
|-------|-----------------|-------------|---------------------|-----------------|
| 1 | 455 | 2.54 | 2.17 | 1.93 |
| 2 | PAVLOV | 2.54 | 2.16 | 1.94 |
| 3 | 453 | 2.54 | 2.14 | 1.88 |
| 4 | 456 | 2.61 | 2.18 | 1.94 |
| 5 | PAVLOV | 2.61 | 2.12 | 1.83 |
| 6 | 566 | 2.60 | 2.20 | 1.73 |
| 7 | PAVLOV | 2.43 | 2.12 | 1.87 |
| 8 | 664 | 2.53 | 2.17 | 1.85 |
| 9 | 353 | 2.58 | 2.16 | 1.95 |
| 10 | PAVLOV | 2.53 | 2.11 | 1.85 |
| 11 | 355 | 2.55 | 2.19 | 1.90 |
| 12 | 436 | 2.42 | 2.13 | 1.93 |
| 13 | 554 | 2.53 | 2.14 | 1.81 |
| 14 | PAVLOV | 2.69 | 2.16 | 1.77 |
| 15 | PAVLOV | 2.49 | 2.10 | 1.75 |
| 16 | 555 | 2.71 | 2.16 | 1.91 |

Table 5.4a - First Phase: Groups 1 to 16

| Group | Winner's Number | Avg. Payoff | Group's Avg. Payoff | Smallest payoff |
|-------|-----------------|-------------|---------------------|-----------------|
| 17 | PAVLOV | 2.48 | 2.11 | 1.75 |
| 18 | 546 | 2.66 | 2.18 | 1.90 |
| 19 | 454 | 2.59 | 2.18 | 1.95 |
| 20 | 515 | 2.33 | 2.14 | 1.96 |
| 21 | PAVLOV | 2.51 | 2.10 | 1.65 |
| 22 | 745 | 2.61 | 2.16 | 1.69 |
| 23 | PAVLOV | 2.51 | 2.14 | 1.90 |
| 24 | 476 | 2.58 | 2.16 | 1.54 |
| 25 | 356 | 2.56 | 2.15 | 1.89 |
| 26 | PAVLOV | 2.50 | 2.10 | 1.84 |
| 27 | 354 | 2.54 | 2.17 | 1.94 |
| 28 | 446 | 2.69 | 2.17 | 1.98 |
| 29 | PAVLOV | 2.63 | 2.08 | 1.79 |
| 30 | 556 | 2.60 | 2.15 | 1.71 |
| 31 | 553 | 2.54 | 2.15 | 1.84 |
| 32 | PAVLOV | 2.83 | 2.14 | 1.92 |

Table 5.4b - First Phase: Groups 17 to 32

| Group | Winner | Winner's Avg. Payoff | Group's Avg. Payoff | Smallest payoff |
|-------|--------|----------------------|---------------------|-----------------|
| 1 | TFT | 2.24 | 1.76 | 1.42 |
| 2 | 556 | 2.21 | 1.92 | 1.31 |
| 3 | TFT | 2.10 | 1.83 | 1.19 |
| 4 | 554 | 2.22 | 1.86 | 1.29 |
| 5 | TFT | 2.18 | 1.80 | 1.47 |
| 6 | TFT | 2.16 | 1.86 | 1.35 |
| 7 | TFT | 1.94 | 1.70 | 1.14 |
| 8 | TFT | 2.10 | 1.75 | 1.25 |
| 9 | TFT | 2.15 | 1.82 | 1.26 |
| 10 | TFT | 2.01 | 1.72 | 1.34 |
| 11 | 445 | 2.39 | 2.05 | 1.41 |

Table 5.5 - Second Phase

| Group | Winner | Winner's Avg. Payoff | Group's Avg. Payoff | Smallest payoff |
|-------|--------|----------------------|---------------------|-----------------|
| 1 | TFT | 2.09 | 1.68 | 1.35 |
| 2 | TFT | 2.23 | 1.66 | 1.17 |
| 3 | TFT | 2.18 | 1.63 | 1.28 |
| 4 | TFT | 2.08 | 1.62 | 1.17 |
| 5 | TFT | 1.76 | 1.56 | 1.05 |

Table 5.6 - Third Phase

| <i>Winner (Player's No.)</i> | <i>Winner's Average Payoff</i> | <i>Group's Average Payoff</i> | <i>Smaller payoff</i> |
|--------------------------------------|------------------------------------|-----------------------------------|-----------------------|
| TFT | 1.51 | 1.36 | 0.78 |

Table 5.7 - Fourth Phase (Final)

5.7 - Discussion of the Results

Except in the final phase, all groups played two simulation rounds. As expected, the ranking order for each group showed variations, because the payoffs accumulated during the iterations are dependent of the sequential random selection of pairs. The tables depicting the results refer to the round in which the greatest average payoff was obtained.

An extra round involving all the 512 fuzzy and the 3 non-fuzzy players in a sole group was also performed. However, though the group played one million iterations, the results obtained were not conclusive, because the participants played only about 7.5 times with each particular opponent, on the average. This small number of iterations between the same pair is not enough for a player to reach a stabilized pattern of behavior with his opponents⁷. The winner of this round was the fuzzy player 555 (the fuzzy "all-D"). TFT ranked 237th; TFT-g and PAVLOV finished in 317th and 374th places respectively. An interesting feature observed was that the 9 first contestants had "5" (which means defection in every circumstance) as the middle digit. This position corresponds to the factor f_2 and refers to the decision rule regarding the payoffs previously obtained. In the limited experiments performed, the "all D" characteristic had an edge over the other players, possibly because it exploits their "nicer" decision rules.

The first phase

In the 64 (2×32) runs, the digit 5 appeared at least once in 36 of the 64 winners. In 28 cases it was the middle digit. Also, in 15 groups, the same fuzzy player won both runs,

⁷ A 486SX 33MHz computer was employed, and the program run in a virtual disk during about 100 hours uninterruptedly, or more than four days.

and 13 of them contained the digit “5” at least once in some position. Excluding the fuzzy players, PAVLOV performed quite well in this phase, winning 14 times. TFT arrived first 3 times.

The second phase

From the 32 groups, 176 fuzzy players have been picked, forming 11 newly arranged sets. The criteria was to select the individuals which appeared simultaneously among the eight best ranked fuzzy players in both rounds, along with some other players that, although not having this feature, also performed well. This time, TFT won 17 of the 22 rounds. Surprisingly, the PAVLOV strategist abandoned the success it had in the first phase and scored poorly. Its best positions were a second and fourth place, ranking in the lower half of the groups most of the times. The explanation for this fact seems to lie in the cognitive rule adopted by PAVLOV, which interprets an opponent’s action as only totally cooperative or defective, though the fuzzy players seldom, if ever, come out with the extreme actions 0 or 1. Those circumstances entailed many C’s from PAVLOV, against less cooperative decisions from its adversaries, and consequent lower payoffs. Why, then, was PAVLOV so successful in the first phase? The answer may be the fact that the “DEFECTION biased” strategists were scattered among all the groups, and in the end they became present in a larger concentration, since they emerged as winners from the preceding level of the tournament. On the other hand, TFT retains a tradition of being good in handling defectors, definitely reciprocating with *zero* whenever it meets another player with a tendency of deciding towards defection, even if it is only a mild disposition. As in the second phase the players with a defective mood outnumbered those more cooperative, TFT thrived, because in a pair of actions, the one nearest to *zero* receives the greatest payoff.

The third and final phases

As can be seen from the respective tables, the third and final phases turned out to be only a confirmation of the tendency that had already been initiated by the increasing exclusion of the “nicer” rules. Nevertheless, a fiercer dispute really has taken place in the last phases. This observation can be confirmed by the average payoffs achieved by the

players, which steadily drop as the tournament evolved. This assertion is valid for both winners and losers. Again, TFT ended in the first position, though closely followed by the 555 fuzzy strategist.

5.8 - Conclusions

The purpose of the model of the IPD presented in this Chapter of the dissertation was essentially the introduction of a fuzzy model of the IPD. Departing from the classical approach where only *all or nothing* decisions could be implemented, the conflict of interest embedded in the game is treated as a problem allowing a continuous interval of strategies, represented by membership grades in the fuzzy sets COOPERATION and DEFECTION which now characterize the players' actions. Additionally, the participants' decisions were guided by fuzzy expert systems that took into consideration other variables besides those related to the history of previous iterations, which are usually employed to assist the strategists. In this framework, a computer tournament was performed, and some results achieved were shortly discussed. Given the importance of the concepts introduced, the interest in further extending the investigation of the FIPD is explored in Chapter 6 of this dissertation, where a subsequent theoretical development as well as a practical application of the paradigm are implemented.

Chapter 6

An Application of the Fuzzy Iterated Prisoner's Dilemma

6.1 - Introduction

In chapter 5, a theoretical model incorporating fuzzy expert systems as a basis for players making decisions in the Iterated Prisoner's Dilemma (IPD) games was presented, simulated and analyzed. As could be seen, allowing the contenders' actions to depart from the traditional Boolean resolutions COOPERATE and DEFECT brings new and important modifications in the game. Moreover, the fuzzy logic approach employed to describe the way the players reason and perceive the stimuli from other players and also from the environment is much nearer real life situations than the former schema. Another significant feature included in the model was to take into account a diversity of variables, other than the usually employed history of previous iterations. The computational tournaments performed using these paradigms brought new light on the manner rational agents might behave when submitted to situations that have a resemblance with the conflict of interests mirrored by the Iterated Prisoner's Dilemma Game.

In this part of the dissertation, many of the ideas extracted from the theoretical developments presented in chapters 4 and 5 will be exploited. The aim is to demonstrate how the structure of a Prisoner's Dilemma game can be applied to a practical problem in the domain of Industrial Engineering, where an evident conflict of interest is present and where the decisions are made by rational agents. Uncertainty, incomplete and/or ambiguous information, risk and subjective perceptions will be still extant in the material that follows. But in the realm of game theory, differently from standard optimization and decision problems, it is possible to combine random events, or states of the world, with actions that are entirely governed by the participants. This characteristic usually brings new difficulties for the decision makers in achieving their desired goals, mainly because a

player's action can always be opposed by a counteraction from the other part, and everyone is simultaneously trying to achieve their own specific goals, which can be opposing (but not always strictly contradictory, though this is not seldom the case).

The Prisoner's Dilemma (PD) is an adequate paradigm to model and understand what happens when rational agents are exposed to those kind of circumstances. It reproduces the contradictions that arise from instances taken from everyday life where the simultaneous search of the individual optimum frequently leads to both individual and collective poor outcomes¹. However, given the qualifications of the problem to be approached, an alternative structure, which partially deviates from the traditional definition of the Prisoner's Dilemma Game, will be employed here. Instead of considering a *two-sided* PD, an *one-sided*² version [RASM89], allied to a fuzzy decision process by one of type of players shall be used to model the problem, which consists of a *Market Share Game*. A detailed method is thoroughly developed, preceded by a description and discussion of the approached problem. A methodology overview, including a simplified outline of the proposed system, is also included. Along with the detailing, an example is supplied, applying the concepts just introduced. The current Chapter ends with a summary of the algorithm.

The information about the computational simulation process and the results obtained is deferred to Chapter 7.

6.2 - Description of the Problem

Among the most important questions that must be tackled by a profit-seeking company (the Firm) operating in a competitive environment, is the estimation of its revenues.

More often than not, the emphasis of the decision process is concentrated in internal aspects of the Firm. These aspects, in general, deal with topics which regard mainly production planning and control, financial management, quality control of products,

¹ This paradox is the essence of the Prisoner's Dilemma Game. For an enlightening discussion of the theme, see [HOFS85].

² See Chapter 3 of the dissertation for an explanation of that class of PD's.

technology employed, investments, relationship with suppliers, sales policy, advertising budget, etc. But a crucial factor influencing a company's performance with respect to its ability to generate profits is the market share it detains. The market share is intimately related to the internally manageable variables mentioned above. Nevertheless, an accurate measure of the impact or effects that projected or just implemented alterations in the attributes of a product, such as quality, price and advertising, will have on the market share is still hard to determine.

The analytical models, generally based on empirical data, constitute an attempt to solve the problem. Unfortunately, the static models suffer from the same plague that beset other economic-related questions: the highly dynamic character of the process and the capacity of evolutionary self-adjustment of the agents involved. This often causes an unsatisfactory performance of traditional static or analytical models, which may turn out to be become either mathematically intractable or unattractive under the point-of-view of the intended precision. In other words, because of its implicit nature, market share problems belong to a class of issues that do not yield easily, in terms of the reliability of results, to standard treatments, such as analytical or even econometric models of prediction.

Computational simulation methods provide an excellent alternative to approach the problem of market share determination. Because of their iterative capabilities, they are able to capture the dynamic and interrelated processes that permeate these kind of questions.

The problem considered in this dissertation is an application of the Fuzzy Iterated Prisoner's Dilemma (FIPD) to a practical situation. It allows the acquisition of new insights of the processes that take place in a competitive market formed by suppliers of products or services and Consumers. The main general questions to be addressed by the model are:

- What are the relationships between the market share of a Firm's product or service and the price, quality and advertising budget the company assigns to it?
- How do the Consumers perceive changes in the attributes of a product or service, and update their decision functions?

- Given the Consumers are diverse regarding their buying power, feature orientation and sensitivity to price and quality, how can the competitors identify market opportunities (niches) and conveniently explore them?
- What is the dynamic behavior of the whole environment, concerning the conflict of interest embedded in the process, both among competitors (sellers) and among Consumers and Firms?
- Can an equilibrium be established? If it can, which are the characteristics of the strategies that compound it?
- Which are the dominant strategies in terms of attaining the best results (profits) for the Firms? Are they stable?

Those questions, along with other additional related analysis, will be the object of a simulated repeated non-cooperative game between Firms and Consumers, both taken as individuals. The iterations will always be pairwise, but the results achieved by both sides will also influence their decisions regarding different contenders. The main goal of this application is the determination of a Firm's market share when operating in a competitive environment, and having rational agents as customers.

6.3 - Methodology Overview

6.3.1 - The Basic Game

The Fuzzy Iterated Prisoner's Dilemma paradigm, in its one-sided version, will serve as the basis for modeling the game. It is considered suitable for the investigation and resolution of the problems posed by the previous questions in the sense that in the relations between a Firm and a Consumer, both sides can COOPERATE or DEFECT.

In the case of a Firm, the degrees of cooperation or defection are expressed by the compromise between the price and the quality of the item offered in the market for sale. Under a Consumer's point-of-view, a Firms' decisions may range from one extreme, denoted by *price* = LOWEST and *quality* = BEST (COOPERATE), to the other, *price* = HIGHEST and *quality* = WORST (DEFECT). On the Consumer's turn, it will go

through a fuzzy-based decision process, but the possible moves will be still dichotomic and expressed by the decision to buy or not the product or service³ when the opportunity to do so appears. While the COOPERATE move implemented by the Consumer means that the transaction has been effectuated, DEFECT implies that it will have to search for another supplier.

Differently from the classic two-sided PD game, in which both players possess the same strategy set and symmetric payoffs, the one sided PD to be employed here does not fit the usual definition of a PD, because the strategy sets for each participant are different and because neither the payoffs nor information sets available to the players are symmetric.

In order to better clarify the mechanisms that will be used to model the proposed application with the one-sided Fuzzy Iterated Prisoner's Dilemma, let us consider for a while, as a simplification, that only the dichotomic punctual choices COOPERATE and DEFECT are allowed for both players, as in the standard game. In this hypotheses, given a typical iteration between any pair of players involved in an iteration, only the four usual feasible outcomes CC, CD, DC and DD may result, regarding the combination of the decisions from the Consumer (potential buyer) and the Firm (potential seller).

The simplified qualitative one-sided PD used in this example is an adapted version of the one introduced by Rasmusen⁴. It can be represented, in its normal form, by the payoff matrix shown in Figure 6.1. The payoffs for the Firm are expressed qualitatively by the terms *high* (HIG), *medium* (MED), *LOW* and *lowest* (LWT), in decreasing ordinal ranking order. On the other hand, the Consumers' gains are reduced to only three possibilities: *high* (HIG), *neutral* (NTL) and *LOW*. This occurs because when the Consumer does not buy the

³ It could be the case that the consumer, too, might be able to decide in a continuous interval, instead of only the two punctual choices *buy* or *not buy*. This would happen if the product offered by the supplier is divisible and the consumer were to select a certain quantity in a given iteration. Although of interest, this relaxation will not be approached in the present work, and is left as a suggestion of future developments. See Chapter 8 - Conclusions.

⁴ Op. Cit. [RASM89]. The difference between the version employed here and Rasmusen's is the existence, in the former, of a row player's *weakly dominant strategy* and the consequent *iterated dominant strategy equilibrium*, not present in the current game.

product or service, the quality of the item is unobserved, and its payoff will always be *neutral* (NTL) in this case. Also, the payoffs are not interpersonal comparable⁵.

| | | | |
|------|---|--|---|
| | | Consumer | |
| | | C | D |
| Firm | C | $\begin{bmatrix} \text{MED, HIG} & \text{LWT, NTL} \\ \text{HIG, LOW} & \text{LOW, NTL} \end{bmatrix}$ | |
| | D | | |

Figure 6.1 - Normal Form of the Simplified One-sided PD with Qualitative Payoffs

In the matrix cells, the first term refers to the row player. The criteria for assigning the qualitative payoffs shown in fig. 6.1 is explained below. They are always taken as a function of the pair of moves selected by the contestants. The first letter, C or D, refers to the row player, the Firm, and the second stands for the column player.

- **CC:** The Firm cooperated in the sense that is offering a product or service of suitable quality at a fair price. The Consumer also cooperated by means of carrying out its decision to purchase from the Firm involved in the iteration. In this situation, the Firm receives a reasonable, or *medium* (MED) payoff, because it accomplished the sale, but the profit margin embedded in the transaction is not too high. This result derives from the assumption that production costs of good quality products are higher. On the other hand, the Consumer gets a *high* (HIG) payoff, for having succeeded in achieving a product or service of suitable quality, at a convenient price.

⁵ Interpersonal comparison of utilities is one of the crucial problems of Game Theory. Some theorists, e.g. [BACH77] admit that it is not possible at all. Fortunately, in the present model, this feature is not required.

- **CD:** The Firm's behavior is the same as in the previous case, but now the Consumer opts for not taking the offer. The Firm's payoff is the *lowest* (LWT), as long as it is not being compensated (consummate the sale) by the Consumer for its efforts in offering it a good deal, in the Firm's point-of-view. The buyer, having refused to purchase from this Firm and possibly having lost an opportunity to achieve a good valued item, will receive a *neutral* (NTL) payoff.
- **DC:** The Firm's article is of inferior quality and expensively tagged, yet the Consumer buys it. As a consequence, the Firm gets a *high* (HIG) payoff, while the other player ends up with LOW, for evident reasons.
- **DD:** The article's attributes are as above, unfavorable to the prospective buyer, and it decides to refuse the deal. Regarding the seller, its gain will be almost as low as in a CD outcome, but because the efforts involved in offering a poor quality and high priced item are much lesser than in the latter case, the loss will be somewhat smaller. For this reason, a *low* payoff will be assigned to it. The Consumer is also in a position similar to the one that happens in CD, however it must be satisfied for having avoided the worst, with its refusal to accomplish an adverse acquisition. Recalling the features possessed by the one-sided PD⁶, it is important to mention that only the row player is confronted with a dilemma equivalent to that which takes place in the standard two-sided PD. This is so because in the one-sided PD at least one of the players prefers CC mostly.

Though in the qualitative payoff matrix both the Firm and the Consumer have identically labeled gains with ordinal equivalence, they do not stand for equal cardinal values. In the one-sided PD, this asymmetry is the main relaxation of the rules that define the usual two-sided PD. Neither player has a dominant strategy, and so there is no

⁶ See Chapter 3.

dominant strategy equilibrium. Nevertheless, a unique DD Nash equilibrium persists. A summary of the players' actions and associated gains are presented in Table 6.1.

| A c t i o n s | P l a y e r s | |
|---------------|---|---|
| | CONSUMER | FIRM |
| COOPERATE (C) | After meeting a Firm, according to a specific selection mechanism, the Consumer decides to purchase the product or service being offered, also conforming to a particular criteria. The payoff it receives is <i>high</i> or <i>low</i> , depending on whether the purveyor cooperated or defected. | Independently of being chosen or not, the policy adopted by the Firm is to offer products or services with good quality at fair prices. Its payoff in iteration is <i>medium</i> or <i>lowest</i> , if the Consumer cooperated or defected, respectively. |
| DEFECT(D) | After choosing a Firm, according to a specific selection mechanism, the Consumer opts for NOT to buy from this supplier, and will have to look for another one. The payoff is <i>neutral</i> (zero). | Independently of being chosen or not, the policy adopted by the Firm is to offer poor quality and expensively priced products or services aiming a high unitary profit. The payoff it receives is <i>high</i> or <i>low</i> , if the Consumer cooperated or defected, respectively. |

Table 6.1 - Actions and Payoffs in a Simplified One-sided Market Share PD

6.3.2 - Basic Iteration Process

In order to implement the model, a hypothetical setting is initialized, and a simulation program will be run to generate the iterations. The components of the process are summarized in Table 6.2.

A Market Share Game

Players

A small number of profit-seeking Firms (sellers) and a population with a large number of Consumers (buyers). In every iteration, both players are chosen randomly, according to specific probability distributions.

Information

Imperfect, Complete, Certain and Asymmetric

Actions and Events

The game is initialized with m Firms and n Consumers. Each Firm offers a similar product or service in the market and makes introductory decisions about its advertising budget, price and quality, that can be altered during the course of the game. All Consumers have two parameters regarding their sensitivity to price and quality. These two attributes affect the way an individual perceives and values an item. Based on its perception, the Consumer decides to buy or not. In either case, payoffs are assigned to the players.

Payoffs

Asymmetric, numerical. For the Consumer, a function of the item's attributes, its decision and individual characteristics. For the Firm, either the item's unitary profit (price minus cost) or a punishment, which will vary according to the iteration's outcome and item's features.

Table 6.2 Main Components of the Market Share Game

6.4 - Detailed Formulation of the One-sided PD Market Share Game

6.4.1- The Sellers (Firms)

In a competitive market, a small number of sellers offer a similar product or service for sale to a population of Consumers. The number m of Firms present in the game is arbitrary. In this application, the simulations will be run with $m \leq 6$, but this number can be altered. The products or services offered by the Firms are differentiated from the Consumers' point of view, by their respective prices p_{ig} , and qualities q_{ig} , $i = 1, 2, \dots, m$, $g = 1, 2, \dots, k$ ⁷. The quality and cost of an item are closely interrelated, so that each of these variables can be expressed as an increasing function of the other⁸. These variables are entirely governed by a seller's decisions, and are perceived in distinct ways by the Consumers. When a pairing between a Firm i and a Consumer j takes place, the latter distinguishes the price p_{ig} before it makes its decision of buying or not the item. The quality q_{ig} , though, is unobserved by the customer unless it purchases the product or service in perspective.

A third factor, also controlled by the companies, is the advertising budget a_{ig} . While playing an important role in attracting potential customers, the amount spent in publicity does not influence the product quality, but will effectively contribute to the Firm's overall costs. When the game starts, each competitor defines the variables p_i , c_i , and a_i , which can be modified in the course of the simulations depending on the results attained. The sellers' global goal is the maximization of their profits. The profit of a Firm during a given period is a function of those three variables, and also of the quantity of items sold. On its turn, the volume of sales is determined by successful iterations with the customers, symbolized by their *Cooperation* in the one-sided PD, as previously explained. In this version of the model, no restrictions are being imposed on the production capacity of the competitors, taken as unlimited. All these variables will be explained in the sequence.

⁷ The index g refers to the *cycle* of the game. See Sections 6.4.2 and 6.4.3.

⁸ See Section 6.4.4 - *The Firm's Revenues, Costs and Profit*.

6.4.2 - The Advertising Budget

Before the iterations start, along with the price p_{ig} and the quality q_{ig} , each Firm i also defines an advertising budget a_{ig} that remains constant during a fixed period, called *cycle g*, denoted by a specified total number of iterations scheduled to take place in this period of the game. It is assumed that the Firms know the size of the potential market, I , and a_{ig} is a number between zero and one, taken as a percentage of the maximum theoretical possible revenues that would be conjointly achieved by all Firms in the limiting hypothesis as explained in section 6.4.4. Regarding only the random selection process of the Firms, the total absolute value allocated to the publicity of the items being sold by each Company is not relevant in this aspect of the simulations performed⁹. Instead, the relation between the amounts are considered.

The publicity expenses have an impact on the sellers' overall costs, and the parameter that will be used to account for this fraction of the costs will be the maximum price of an item considered in the game, $\max p_{ig}$, arbitrarily fixed in 10 monetary units, as will be further made clear.

The selective process of a particular Firm to take part in an iteration is random, according to a probability distribution derived from the various advertising budgets a_{ig} . The probability s_{ig} of a supplier i to be selected during a cycle g will be given by:

$$s_{ig} = \frac{s}{m} + \frac{a_{ig}}{\sum_{i=1}^m a_{ig}} \times (1 - s), \quad \text{Eq. 6.1}$$

The idea that underlies the formulation of s_{ig} by Equation 6.1 is that any Firm can be picked by the random selection process with a probability $\frac{s}{m}$ even though it refrained from spending in advertising. In other words, in the hypothetical market there will always be a probability s , arbitrarily fixed at the beginning of a cycle g , independent of the Firms' advertising budgets a_{ig} . That probability will be shared by all the m competitors, that a Firm interacts with a consumer. Obviously, the more a seller spends in publicity, mirrored by the

⁹ However, the absolute value of the advertising budget chosen by the suppliers is consequential in the task of establishing the net profit of a Firm during a cycle of the game. See section 6.4.4.

value attributed to a_{ig} , the greater are its chances of being chosen more times as a potential supplier, but that advertising effort will actuate only on the remaining probability $(1 - s)$.

As mentioned before, the cost incurred in advertising does not have any efficacy in improving the quality of the product or service and consequently a Consumer's predisposition in purchasing it, but this decision will affect the seller's total profit achieved at the end of each cycle¹⁰. The profit can be positively or negatively influenced, as long as there is a trade off between the capacity of pulling in a larger number of potential customers, mirrored by an enhanced chance of a Firm being selected when s_i grows, and the increased costs when a_{ig} is raised. Also note that the efficacy of a_{ig} is relative, not absolute. In this manner, if every Firm elect *exactly* the same value for its advertising expenses in a given cycle, its expected number of iterations will be the same as if none had spent anything¹¹.

6.4.3 - The Size of the Potential Market

The global game will be partitioned in k *cycles* of identical length. The length of a cycle is defined by an elective expected number of iterations per Consumer present in the game. When a cycle is completed, the Firms will be allowed to modify their previous decisions regarding the price, quality and the advertising budget. The criteria used to alter those variables is the pursue of optimum economic results, namely, the profit.

The size I of the potential market of the product or service is constant during each cycle of the game. It is given by the totality of Consumers (n) times the expected quantity of purchase opportunities per Consumer $E(b)$ per cycle. This means that I random pairings will occur between sellers and buyers. It is important to recall that a pairing does not necessarily mean that a transaction was completed, what only occurs if the Consumer *cooperates*, or decides to accomplish the purchase. In every cycle g , the number of times a

¹⁰ See next section for the definition of this parameter.

¹¹ To avoid a *division by zero* error in the simulation program, that hypothesis will be withdraw from the options to be run.

particular Consumer j , $j = 1, 2, \dots, n$ interacts with a Firm i , $i = 1, 2, \dots, m$, will be defined as b_{igj} , and the absolute frequency of *cooperations* per Consumer shall be called \bar{b}_{igj} .

$$I = \sum_{j=1}^n \sum_{i=1}^m b_{igj} = n \times E(b) , \forall g \quad \text{Eq. 6.2}$$

The quantity of items sold by the Firm i during a cycle is v_{ig} , and the aggregated effective demand for all sellers is V_g .

$$v_{ig} = \sum_{j=1}^n \bar{b}_{igj} \quad \text{Eq. 6.3}$$

$$V_g = \sum_{j=1}^n \sum_{i=1}^m \bar{b}_{igj} \quad \text{Eq. 6.4}$$

6.4.4 - The Firms' Revenues, Costs and Profit

1. Revenues

Each time a Firm meets a Consumer, basically only two things can happen: either the customer purchases the item offered (*cooperates*) or declines (*defects*). On the other hand, at the beginning of every cycle the seller has a continuum of options¹² regarding its policies towards the combination of price and cost of its product or service.

The revenues that a Firm receives in the course of every cycle of the game are originated from its payoff function. The inputs of that function are the moves performed by the Firm and the other player. If the Consumer *cooperates*, the payoff r_{ig} to the Firm shall be defined as the unitary profit ($p_{ig} - c_{ig}$). Otherwise, the Firm is penalized by receiving a negative payoff. Here, the *penalty* payoff was discretionary fixed, consisting of

¹² The choice of prices, costs and advertising budgets for each Firm is made by the researcher.

two fixed parts t_1 , t_2 and a portion proportional to the ratio $\frac{c_{ig}}{p_{ig}}$, which can be interpreted as a Firm's degree of cooperation.

| Firms' Payoffs | Consumer's Decision | |
|----------------|----------------------------|---|
| | <i>Cooperates (buys)</i> | <i>Defects (does not buy)</i> |
| | $r_{ig} = p_{ig} - c_{ig}$ | $r_{ig} = - (t_1 + t_2 \times \frac{c_{ig}}{p_{ig}})$ |

Table 6.3 - The Firms' Payoffs

This scheme is consistent with the assumption that the loss associated with an eventual non-consummated sale is greater when the Firm *cooperated* in a larger degree, which means producing and selling an item with a lower unitary profit margin. Defection, though capable of bringing better short-term benefits to the Firm, also has its risks, since in the long run it will generally drive away would-be buyers, as will be explained in section 6.4.6.

The introduction of the subscript g to represent a particular cycle of the game is necessary because the Firms can alter the variables p_{ig} , c_{ig} and a_{ig} when a new cycle g begins. This possible modification in the sales policy can bring different results in terms of the revenues achieved, and shall constitute a subject of interest for the analysis of the simulations.

Concerning the variables p_{ig} , c_{ig} , and q_{ig} it is necessary to establish some important rules that must be obeyed by during the game: drive away

- p_{ig} and c_{ig} are expressed in monetary units;
- The quality attribute q_{ig} is directly related to the cost of the item;
- No Firm is allowed to decide on a price that is inferior to the cost of item ($p_{ig} \geq c_{ig}$);

- All prices are subject to a maximum and a minimum, that have been selected as 10 and $(0+\varepsilon)$ monetary units, respectively, for the purposes of this model $((0+\varepsilon) \leq p_{ig} \leq 10)$;
- During a cycle of the game, p_{ig} , q_{ig} and a_{ig} remain constant.

The total gross revenues obtained by a Firm i after the end of a cycle g , $g=1, 2, \dots, k$ is:

$$R_{ig} = (p_{ig} - c_{ig}) \times v_{ig} - (t_1 + t_2 \times \frac{c_{ig}}{p_{ig}}) \times \left[\left(\sum_{j=1}^n b_{jig} \right) - v_{ig} \right] \quad \text{Eq. 6.5}$$

II. Costs and Quality

Two kinds of costs are present in each cycle g of the game:

- a) Unitary production cost c_{ig} , and
- b) Advertising cost a_{ig} .

The total operational cost of a Firm i in a cycle will be a function of its current item's price, cost, the fixed parameter t , the number of iterations in which the Firm took part (that could be called "customers' visits"), the number of consummated and non-consummated sales and the amount spent in publicity.

a) Unitary Costs and Quality

Because of the structure adopted for the simulation model, the operational costs will be considered as embedded in the payoff function of each Firm, and to avoid redundancy, they will not be treated separately. Thus, when selecting p_{ig} and c_{ig} , the Firm is deciding on its own payoff function, and, in a way, also partially on the Consumer's. This occurs because the quality q_{ig} that the buyer assigns to the item is taken into account

in the model as a crescent function of the cost c_{ig} , settled by the seller to the item in a cycle g^{13} , that is, $q_{ig}=f(c_{ig})$.

Note that, as the quality q_{ig} is not observed by the Consumer unless it purchases the item, the information in the game becomes asymmetric. This is so because while the seller knows with anticipation its gains for any move that the other player eventually chooses, the same privilege is not granted to the buyer, whose access to this knowledge is denied beforehand. In this manner, only the advertising costs are considered apart from the payoff function.

According to the basic regulations established for the ranges of c_{ig} and p_{ig} (section 6.4.4 - I), c_{ig} must be less or at most equal to p_{ig} . As will be seen in the sequence, each Consumer, depending on its own sensitivity parameters regarding price and quality, will assess these two variables with the aid of two specific groups of fuzzy sets.

The sensitivity parameters have opposite signals but the same absolute value, and as quality and price have reverse affinities, a mutual cancellation of effects takes place, so the fuzzy linguistic classifications of these attributes are equivalent. Thus, if q_{ig} is taken simply as equal to c_{ig} , the Consumer's perception of the quality of any item would be at most as favorable as the price, in the extreme hypotheses when $c_{ig}=p_{ig}$. This situation, though allowed by the game's rules, is unrealistic, since the Firms must have some unitary margin of profit in their products or services, and they must be seen by the Consumer as appropriate if a deal is to be accomplished. With those considerations in mind, a crescent relation between cost and quality is proposed.

The grounds for the formulation of $q_{ig}=f(c_{ig})$ come from the presupposition that every Consumer will assign a quantitative measure of quality q_{ig} to an item greater than its cost c_{ig} . In this manner, the quality q_{ig} of an item is expressed as a nonlinear function of its

¹³ This assumption is, of course, a great simplification. The list of factors that have influence in the quality of a given item is very extensive and diverse. Operational and administrative efficiency, technology, availability of materials and of a trained labor team, experience, can be cited as some of the less apparent components of quality. Furthermore, there is still the subjective perception of the user. In the more recent specialized literature on marketing, (e.g. [MCM192]) a holistic view of the process is usually employed, where cost and quality are dissociated and sometimes even inversely correlated. Nevertheless, regarding unitary variable production costs, it is generally considered that a strong correlation between cost and quality can be assumed for most kinds of products and services. So, for the sake of tractability, making the variables cost and quality directly interrelated in the model has been heeded as a reasonable approach.

production cost c_{ig} , *plus* an increment, as shown in Equation 6.6. The increment is a fraction of the cost c_{ig} , depicted in Equation 6.7.

$$q_{ig} = \frac{\left\{ 10 \times \left[3 \times \left(\frac{c_{ig}}{10} \right)^2 - 2 \times \left(\frac{c_{ig}}{10} \right)^3 \right] \right\} + c_{ig}}{1 + \frac{c_{ig}}{10}} \quad \text{Eq. 6.6}$$

$$\text{Increment} = \frac{\left\{ 10 \times \left[3 \times \left(\frac{c_{ig}}{10} \right)^2 - 2 \times \left(\frac{c_{ig}}{10} \right)^3 \right] \right\} + c_{ig}}{1 + \frac{c_{ig}}{10}} - c_{ig} \quad \text{Eq. 6.7}$$

The increment of quality over cost was formulated having as basis an arbitrary polynomial function with a S-shaped curve. The criterion for choosing the parameters of Equation 6.6 was to pick the simplest form that fitted the desired effect (obtaining a S-shaped curve). It was accomplished by using the polynomial $y=3x^2 + 2x^3$ ($y, x \in [0, 1]$) as the starting point. Then, the parameters were mathematically adjusted, so that when the *increment* is added to the second member of the linear equation *quality = cost*, the range of the final function q_{ig} would be inside the same interval of the cost c_{ig} , that is, $[0,10]$. The introduction of an increment with that specific mathematical behavior is based on the assumption that the gradient of quality, though always positive in the interval, is crescent until a certain limit, thereafter decreasing¹⁴.

The hypothesis sketched above mirrors the presumption that the products or services which are of superior quality (and consequently most of the times higher priced) generally require a disproportional larger effort to produce in terms of cost. In other words,

¹⁴ $\frac{\partial^2 q_{ig}}{\partial c_{ig}^2} = 0$ occurs at $c_{ig} = 2.5992$, and the maximum value of the increment is 1.8034 at $c_{ig} = 6.18034$.

the model is assuming that the quality of an item, as judged by the Consumers, increases with the associated cost of production initially at a crescent pace and then at decreasing rates. This conjecture is in accordance with the seemingly willingness of wealthy buyers to pay imbalanced more expensive prices for items with high-end features.

The relation between the cost c_{ig} and the quality q_{ig} (function $q_{ig}=f(c_{ig})$ - Equation 6.6), and the absolute value of the increment (Equation 6.7) are illustrated in Figure 6.2.

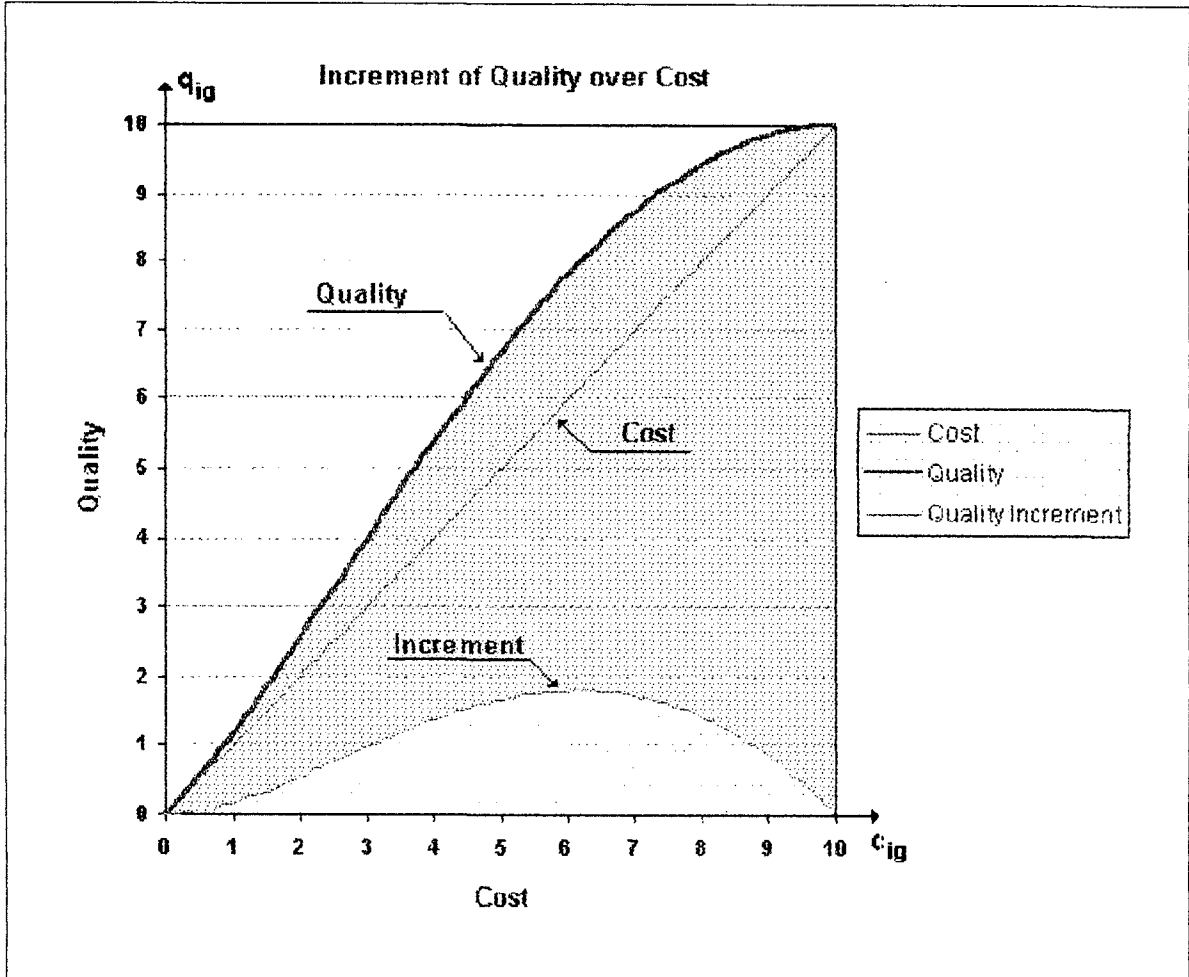


Figure 6.2 - Relation between the Quality q_{ig} and the Cost c_{ig}

b) Advertising Costs

In order to determine the absolute value, in monetary units, of the advertising budget, the following method will be adopted:

i) The total number of potential sales per cycle is I . In the limiting case, the maximum unitary profit ($p_{ig}-c_{ig}$) a Firm i can achieve per item sold in every cycle g is 10 MU. This could happen in the extreme (and highly improbable) hypothesis when $p_{ig} = 10$ and $c_{ig} = 0$. If $p_{ig} - c_{ig} = 10$ for all i , then the maximum theoretically attainable total operational profit is $10 \times I$ for every cycle g .

ii) Using the limit amount $10 \times I$ as a reference, the absolute value A_{ig} allocated to the advertising expenses by a Firm i in a cycle g shall be represented by

$$A_{ig} = a_{ig} \times 10 \times I . \quad \text{Eq. 6.8}$$

III. Profit

Each Firm's profit P_{ig} will be computed at the end of each cycle g . It is expressed simply by the difference between the algebraic sum R_{ig} of the payoffs received, or the revenues, and the advertising expenses.

$$P_{ig} = R_{ig} - A_{ig} \quad \text{Eq. 6.9}$$

The profit achieved by a competitor in each cycle shall be the most important factor that a Firm can use to revise its decisions regarding the policy to be adopted in later cycles. It must be recalled that the sellers are considered rational agents, and will act accordingly in the pursuit of the best possible results.

6.4.5 - The Consumers

The population of buyers interacts with the Firms with the objective of purchasing the product or service that they offer in the market. As it was already disclosed, the Consumers are not all alike. They differ regarding their preference functions regarding the price and the quality of the item in perspective. This is equivalent to saying that they have distinct utility functions regarding these attributes, and they will act to maximize it. The individualization of each Consumer taking into account all the plausible factors that contribute to each own utility function is a very hard task. Nevertheless, regarding the

buying decision of similar purpose items, it can be said that the vast majority of individuals would give preference to achieve the best possible value for the money spent. It is also reasonable to assume that the income (or wealth, buying power) has a decisive role in the profile of preferences of a Consumer.

With those considerations in mind, it is proposed here that the Consumers be characterized by their income or buying power, which by its turn will affect their respective sensitivities to price and quality of a product or service, and generate the rules that they will apply to make their acquisitions.

1. The Frequency Distribution of the Consumers' Income

The population of Consumers will consist of 1000 individuals. They are classified, according to their income, in ten classes, approximately following a *Beta* probability density function, arbitrarily selected to represent the dispersion of the incomes. The absolute value of this variable does not have influence in the game's process and results. So, all individuals have incomes which are normalized in the interval (0, 1). Note that a buying power nearby zero does not contain any indications about an individual's absolute wealth, or income. All potential customers present in the population are supposed to detain a *minimal wealth*, which is *immaterial* to the purpose of the simulations to be run. What really matters, then, is the *Normalized Income*, which represents the differences among Consumers under the aspect of sensitivity to an item's price and associated quality.

When the game is being played, the selection process of a Consumer from the population is made randomly, according to distribution of Incomes. No two players are exactly alike, since the criteria for generating the wealth of every player was to find one *Normalized Income* for each of the 1000 levels of the accumulated probability, calculated for equal steps of 0.1%. The reason for grouping the players in ten classes is linked to the adequate formatting of data aiming the convenience required for the posterior analysis of the results concerning the Consumers' behavior.

The relative frequency, as well as the limits considered for the classes of the population's *Normalized Incomes* are shown in Table 6.4.

| Customer's Class | Normalized Income (Range) | Relative Frequency (%) | Accumulated Relative Frequency (%) |
|------------------|---------------------------|------------------------|------------------------------------|
| I | 0.00 - 0.10 | 8.0 | 8.0 |
| II | 0.10 - 0.20 | 13.7 | 21.7 |
| III | 0.20 - 0.30 | 15.5 | 37.2 |
| IV | 0.30 - 0.40 | 15.6 | 52.8 |
| V | 0.40 - 0.50 | 14.3 | 67.1 |
| VI | 0.50 - 0.60 | 12.2 | 79.3 |
| VII | 0.60 - 0.70 | 9.6 | 88.9 |
| VIII | 0.70 - 0.80 | 6.6 | 95.5 |
| IX | 0.80 - 0.90 | 3.5 | 99.0 |
| X | 0.90 - 1.00 | 1.0 | 100 |

Table 6.4 - Relative Frequency Distribution of the Consumers' Income

The frequencies shown have been approximated from a probability density function of a Beta Distribution, with parameters $a = 1.6093$ and $b = 2.41395$. A complete tabulation of the Normalized Incomes to be used in the simulations is in Appendix (a). The frequency histogram corresponding to Table 6.4 and the respective Beta density function curve are illustrated in Figure 6.3.

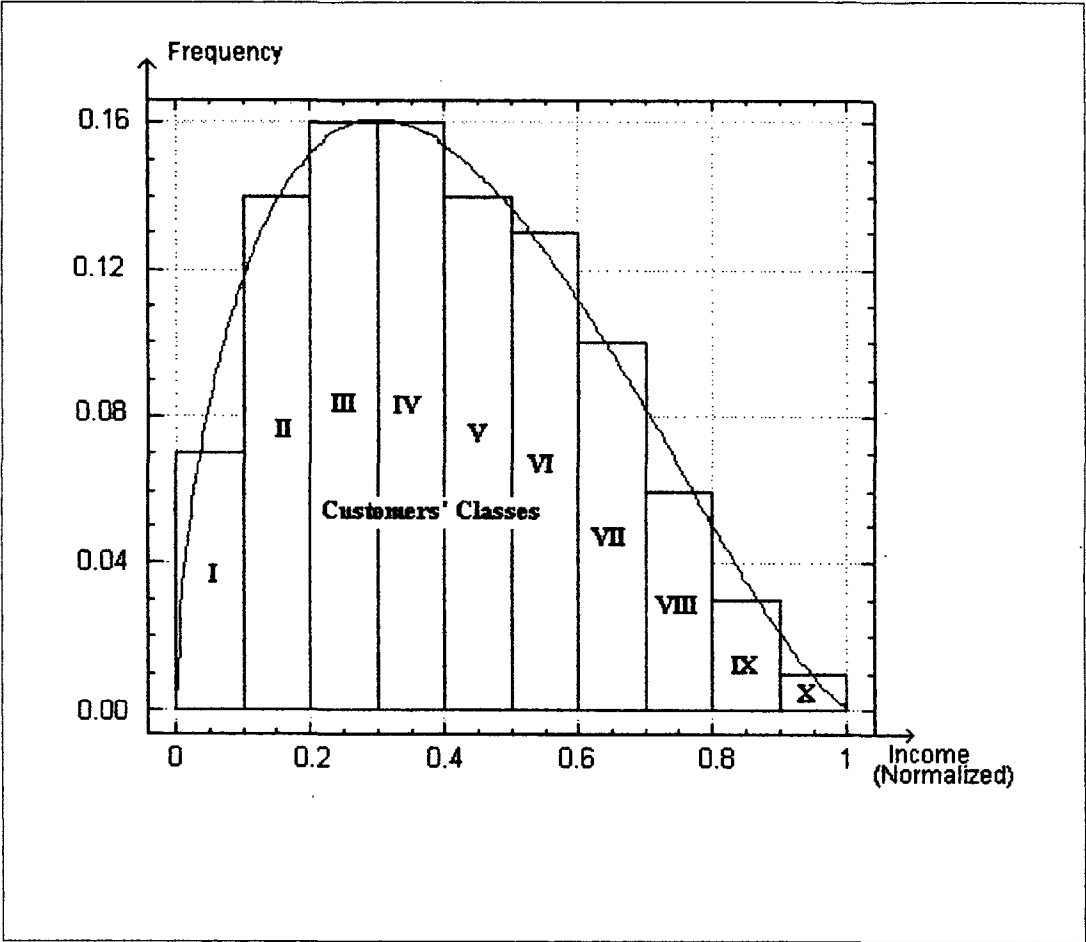


Figure 6.3 -Consumers' Income Frequency Histogram with the Respective Beta Distribution Fitting

II. Characterization of the Consumers' Income by Fuzzy Sets

The universe of discourse of the Consumers' Normalized Income, defined by the interval (0, 1) will be divided into fuzzy sets, which in this case are all either of trapezoidal or triangular shapes. The main motives for this simplification are two: First, the volatility and subjectivity of the criteria for the classification of this kind of data, even when real figures are available, which is not the case in this research model. Second, it is a well acknowledged fact the one of the advantages of the fuzzy logic tools is its remarkable ability to yield very satisfactory results even when the characterization of the data is not precise. In other words, the fine tuning of the shapes of the fuzzy sets responsible for modeling the information should not be a critical point for achieving success in the proper

working of the system built. If the outputs are too sensible to this approach, then maybe another more suitable tool should be found to deal with the problem. The same can be said about the points which define the fuzzy sets. Here, the criteria employed to determine the limiting and central points of the linguistic variables qualifying the Consumers' Income was based on its accumulated frequency distribution f_{ac} .

| Linguistic Classification (Fuzzy Set) | Shape | Designation Points f_{ac} | Normalized Income $w_j = f(f_{ac})$ |
|--|-------------|--------------------------------|--|
| Very Low (VL) | Trapezoidal | 0.00 | 0.00000 |
| | | 0.10 | 0.11609 |
| | | 0.25 | 0.22145 |
| Low (LW) | Triangular | 0.10 | 0.11609 |
| | | 0.30 | 0.25369 |
| | | 0.50 | 0.38140 |
| Medium (MD) | Triangular | 0.30 | 0.25369 |
| | | 0.50 | 0.38140 |
| | | 0.70 | 0.52183 |
| High (HG) | Triangular | 0.50 | 0.38140 |
| | | 0.70 | 0.52183 |
| | | 0.90 | 0.71354 |
| Very High (VH) | Trapezoidal | 0.75 | 0.56203 |
| | | 0.90 | 0.71354 |
| | | 1.00 | 1.00000 |

Table 6.5 Specification of the Consumers' Income Fuzzy Sets

The variable $w=f(f_{ac})$ is the Normalized Income, and the argument, f_{ac} , is the accumulated relative frequency. Figure 6.4 depicts the Fuzzy Sets selected to qualify w_j , with the degrees of membership $u(w_j)$ shown in the vertical axis. Five levels of Income have been designed to accommodate the Consumers' wealth.

Why five fuzzy subsets? As will be seen subsequently, the domain of the several variables used in the current model is always partitioned in five linguistic qualifications. The reason for doing so is not grounded on theoretical or empirical evidences, but emulates a

criterion commonly found in the literature for designating the number of membership function labels, which are generally limited to seven (e.g. [INFE91], [MUNA94], [KOSK93], [HALL94], [VIOT94], [DEBO94]). On the other hand, around five is frequently the number used for classifying economic groups. The tuning of fuzzy expert systems in control problems regarding the definition of the fuzzy subsets employed can be achieved by an automated process, e. g. genetic algorithms [NOMU94], [COX95]. However, the model concerns human perceptions regarding a mental decision process, and it does not appear plausible neither that a much larger number of qualifications for the variables involved would be adequate, nor that the agents might implement AI-aided methods.

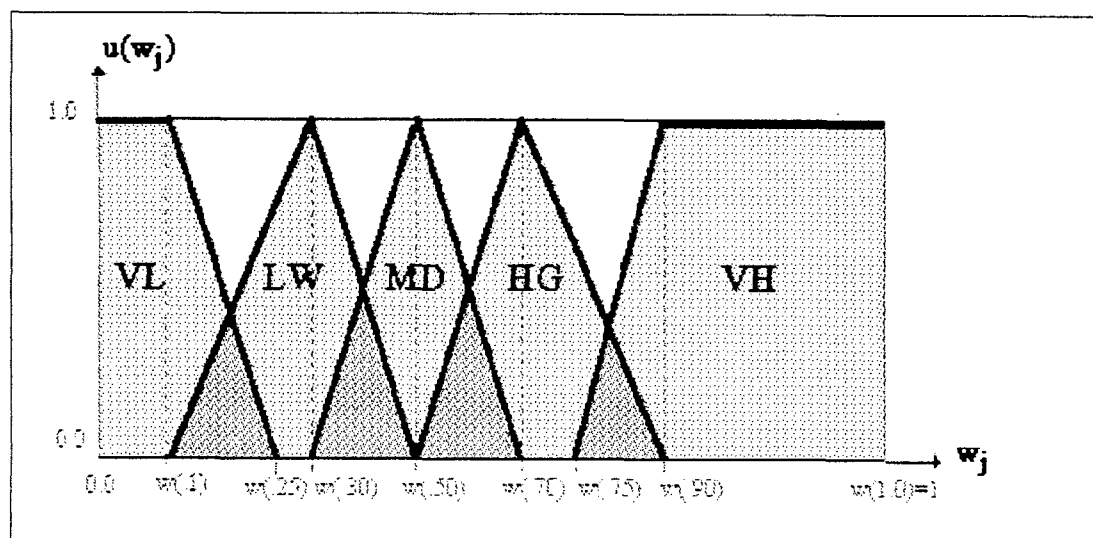


Figure 6.4 - Fuzzy Sets Qualitatively describing the Consumers' Normalized Income $w_j = f(f_{ac})$

Equations 6.10(a) to 6.10(e) determine the degrees of membership of w_j in the fuzzy sets defined by the points f_{ac} chosen.

$$u_{VL}(w_j) = \begin{cases} 1 & \text{if } 0 \leq w_j \leq 0.116 \\ -9.492 w_j + 2.102 & \text{if } 0.116 < w_j \leq 0.221 \\ 0 & \text{if } w_j > 0.221 \end{cases} \quad \text{Eq. 6.10 (a)}$$

$$u_{LW}(w_j) = \begin{cases} 0 & \text{if } 0 \leq w_j \leq 0.116 \\ 7.268 w_j - 0.844 & \text{if } 0.116 < w_j \leq 0.254 \\ -7.830 w_j + 2.986 & \text{if } 0.254 < w_j \leq 0.3813 \\ 0 & \text{if } w_j > 0.3813 \end{cases} \quad \text{Eq. 6.10(b)}$$

$$u_{MD}(w_j) = \begin{cases} 0 & \text{if } 0 \leq w_j \leq 0.254 \\ 7.830 w_j - 1.986 & \text{if } 0.254 < w_j \leq 0.381 \\ -7.121 w_j + 3.716 & \text{if } 0.381 < w_j \leq 0.522 \\ 0 & \text{if } w_j > 0.522 \end{cases} \quad \text{Eq. 6.10 (c)}$$

$$u_{HG}(w) = \begin{cases} 0 & \text{if } 0 \leq w \leq 0.381 \\ -7.121 w - 2.716 & \text{if } 0.381 < w \leq 0.522 \\ -5.216 w + 3.722 & \text{if } 0.522 < w \leq 0.714 \\ 0 & \text{if } w > 0.714 \end{cases} \quad \text{Eq. 6.10(d)}$$

$$u_{vH}(w) = \begin{cases} 0 & \text{if } w < 0.562 \\ 6.600 w - 3.709 & \text{if } 0.562 < w \leq 0.714 \\ 1 & \text{if } w > 0.714 \end{cases} \quad \text{Eq. 6.10 (e)}$$

III. The Consumers' Sensitivities to Price and Quality

Every Consumer is individualized by its Normalized Income, and that attribute will be used to establish *sensitivity parameters* to the price p_{ig} and the quality q_{ig} of a product

or service in prospect. To accomplish this objective, an universe of discourse for those two parameters is established, and qualitatively described by fuzzy sets.

The sensitivity parameters will serve as references to modify the way each buyer perceives the price and the quality of the item being sold. Whereas the price p_{ig} can be appraised by the customer, and solely by it, *before* the decision to buy or not the product or service is made, the complete appreciation of the quality q_{ig} only can be properly consummated *if and after* the Consumer bought the item.

Nevertheless, regarding the quality, the players will be able to have access to some partial information transmitted from the other customers who previously had a concrete experience with products or services purchased from Firm i , with which the present player is considering to deal. This topic of the model will be explored in detail ahead.

The price of a product or service item being sold in the market, when evaluated by any two Consumers with different Normalized Incomes is classified in distinct manners. This seems to be a natural approach, since in realistic situations people with a greater wealth may find some price “reasonable”, while a less affluent individual is likely to assess it as more costly. Thus, the sensitivity parameters that are being introduced in the model have as objective the accounting for those individual differences in perception.

Each sensitivity parameters s_{pj} , s_{qj} represented in the horizontal axis of Figure 6.5 will be determined, for every Consumer, by means of a set of fuzzy production rules, which have as antecedents and consequents the fuzzy sets illustrated by Figures 6.4 and 6.5, respectively.

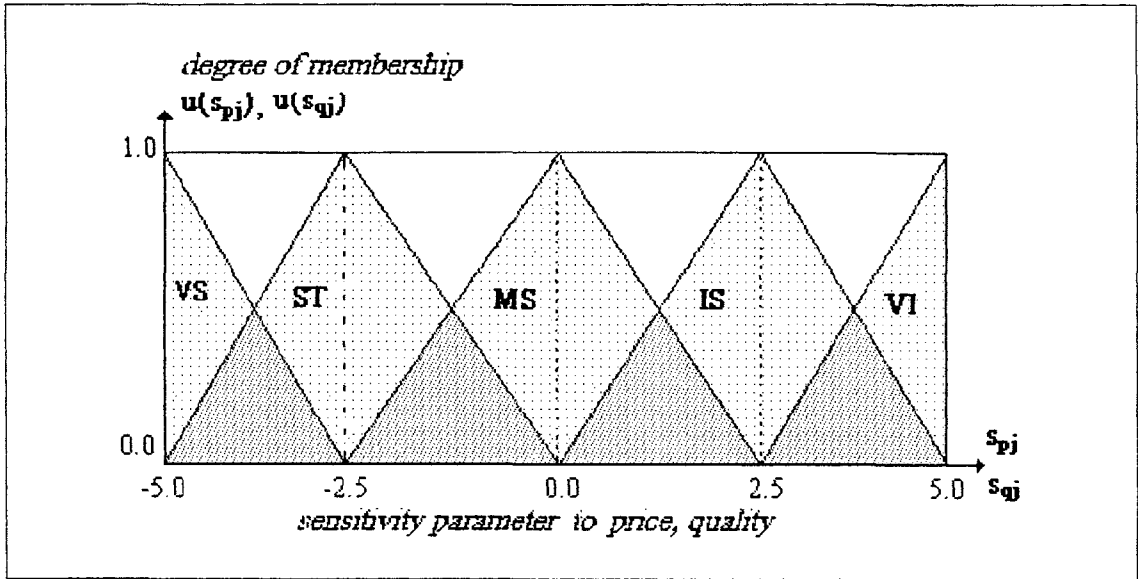


Figure 6.5 - Fuzzy Sets describing the Consumers' Sensitivity to Price and Quality

The values of s_{pj} , s_{qj} are obtained through the defuzzification of the linguistic attributes $u(s_{pj})$, $u(s_{qj})$ of the Consumer's sensitivities to price, quality. The general expression for the defuzzification of s_{pj} is depicted in Equation 6.11.

$$s_{pj} = \frac{A_{VS} \times d_1 + A_{ST} \times 0.25 + A_{MS} \times 0.5 + A_{IS} \times 0.75 + A_{VI} \times d_2}{A_{VS} + A_{ST} + A_{MS} + A_{IS} + A_{VI}} \quad \text{Eq. 6.11}$$

In Equation 6.11 above, the variables A_{VS} , A_{ST} , ..., A_{VI} conform to the areas of the triangles representing the various fuzzy subsets, truncated at the height of the corresponding degrees of membership yielded by the production rules from Table 6.6. The variables d_1 , d_2 , and the numeric values are the distances of the centers of gravity of the several areas to the point $s_{pj} = -5$. The defuzzification method chosen was the *center of gravity of area* (COA), as was explained in Chapter 4.

The other parameter regarding quality, s_{qj} , will be taken as the opposite of s_{pj} , that is, $s_{qj} = -s_{pj}$. The rules listed in Table 6.6 are similar for these both parameters, only the consequents are reversed. This means that a wealthy individual, besides being more tolerant

in relation to price, is also more demanding when it focuses on quality, and conversely. Due to the fact that the labels of the fuzzy sets that designate the sensitivities to both price and quality are the same, the reversal of the fuzzy conclusions became necessary, and the defuzzification will simply yield parameters with identical modules and opposite signals. The reason for this differentiation between q_{ig} and p_{ig} is the criterion chosen to modify each buyer's perceptions of price and quality.

The essential idea that underlies the implementation of the proposed method is to promote modifications in a basic pattern of fuzzy sets that designate the non-fuzzy data pertaining price and quality. The alterations will consist in shifting consistently the reference points that define the fuzzy sets in the horizontal axis *and* promoting specific dilation and concentration operations, in a similar manner to adjectival hedges [ZADE72]. As a result, every Consumer will be particularly characterized, and its decisions regarding the same circumstances may be different from other individuals.

The determination of the sensitivity parameters is done with the support of two similar and simple fuzzy expert systems. The only differences in the production rules are in the consequents, which are symmetrically opposed. Table 6.5 displays the rules for both price and quality sensitivity parameters.

- Rule 1:** *If* Income (w_j) is *Very High* (VH) *then* the sensitivity to price (quality) parameter s_{pj} (s_{qj}) is *Very Insensitive* (VI) (*Very Sensitive* (VS)).
- Rule 2:** *If* Income (w_j) is *High* (HG) *then* the sensitivity to price (quality) parameter s_{pj} (s_{qj}) is *Insensitive* (IS) (*Sensitive* (ST)).
- Rule 3:** *If* Income (w_j) is *Medium* (MD) *then* the sensitivity to price (quality) parameter s_{pj} (s_{qj}) is *Medium Sensitive* (MS).
- Rule 4:** *If* Income (w_j) is *Low* (LW) *then* the sensitivity to price (quality) parameter s_{pj} (s_{qj}) is *Sensitive* (*Insensitive* (IS)).
- Rule 5:** *If* Income (w_j) is *Very Low* (VL) *then* the sensitivity to price (quality) parameter s_{pj} (s_{qj}) is *Very Sensitive* (*Very Insensitive* (VS)).

Table 6.6 Rules used by the Fuzzy Expert System that associates a Consumers' Income w_i and the Sensitivity to Price and Quality Parameters s_{pj} and s_{qj} respectively.

The application of the rules from Table 6.6 will determine the output (consequent) *Sensitivity to price (quality) parameter s_{pj} (s_{qj})* from the input (antecedent) *Income (w_j)* for every Consumer j . Then, the basic configuration of the fuzzy sets that describe both the price p_{ig} and the quality q_{ig} of a product or service, illustrated in Figure 6.6, will be updated. This procedure generates a new specific pattern of fuzzy sets for each Consumer j , with which it will perceive the price being charged for the item by any Firm i during a particular cycle g of the game, and the respective quality, if and after the product or service is purchased.

The alterations promoted by s_{pj} and s_{qj} in the basic patterns are of two kinds. First, all the reference points in the horizontal axis ($u(p_{ig})=0$) will be consistently displaced to the right or left¹⁵, of a distance s_{pj} from their original standard location. With this manipulation,

¹⁵ The displacement of the reference points is made to the right or left, depending on whether the signal of s is positive or negative, respectively.

the resulting group of fuzzy sets are vertically sectioned at either one of the right or left extreme points (0 or 10) of the universe of discourse, losing the outer portion.

Second, the membership functions are adjusted by either a *dilation* or *concentration* operator, which by their turn are also functions of s_{pj} and s_{qj} . The criteria proposed in the process is to dilate or concentrate the membership functions which refer to points in the domain of p_{ig} or q_{ig} lesser than $5 + s_{qj}$ (or $5 + s_{pj}$), with the operators:

$$\text{Dilation:} \quad \delta_j = \frac{1}{1 + \frac{|s_{pj}|}{5}} \quad \text{Eq. 6.12(a)}$$

$$\text{Concentration:} \quad \chi_j = 1 + \frac{|s_{qj}|}{5} \quad \text{Eq. 6.12(b)}$$

The direction of the shifting (left or right) and the application of the hedges δ_j and χ_j will depend on the sensitivity parameters determined for every player (buyer), according to the criteria listed in Table 6.7.

The choice of the denominator of $|s_{pj}|$ was arbitrary, but taking into account the range of values that the resulting expression would yield, and the possible shapes of the ensuing curves.

| | | Operation | | |
|---------|--|-----------|--|---|
| | | Shifting | Dilation | Concentration |
| Price | $ s_{pj} : \Rightarrow$ if $s_{pj} > 0$ | | $u_{dil}(p_{ig}) = [u(p_{ig})]^{\delta_j}$ if $p_{ig} < (5.0 + s_{pj})$ | $u_{con}(p_{ig}) = [u(p_{ig})]^{\lambda_j}$ if $p_{ig} > (5.0 + s_{pj})$ |
| | $ s_{pj} : \Leftarrow$ if $s_{pj} < 0$ | | $u_{dil}(p_{ig}) = [u(p_{ig})]^{\delta_j}$ if $p_{ig} > (5.0 + s_{pj})$ | $u_{con}(p_{ig}) = [u(p_{ig})]^{\lambda_j}$ if $p_{ig} < (5.0 + s_{pj})$ |
| | 0 if $s_{pj} = 0$ | | none | none |
| Quality | $ s_{qj} : \Rightarrow$ if $s_{qj} > 0$ | | $u_{dil}(q_{ig}) = [u(q_{ig})]^{\delta_j}$ if $q_{ig} > (5.0 - s_{qj})$ | $u_{con}(q_{ig}) = [u(q_{ig})]^{\lambda_j}$ if $q_{ig} < (5.0 - s_{qj})$ |
| | $ s_{qj} : \Leftarrow$ if $s_{qj} < 0$ | | $u_{dil}(q_{ig}) = [u(q_{ig})]^{\delta_j}$ if $q_{ig} < (5.0 - s_{qj})$ | $u_{con}(q_{ig}) = [u(q_{ig})]^{\lambda_j}$ if $q_{ig} > (5.0 - s_{qj})$ |
| | 0 if $s_{qj} = 0$ | | none | none |

Table 6.7 - Operations for modifying the Fuzzy Sets that depict the Consumers' Perceptions of Price and Quality

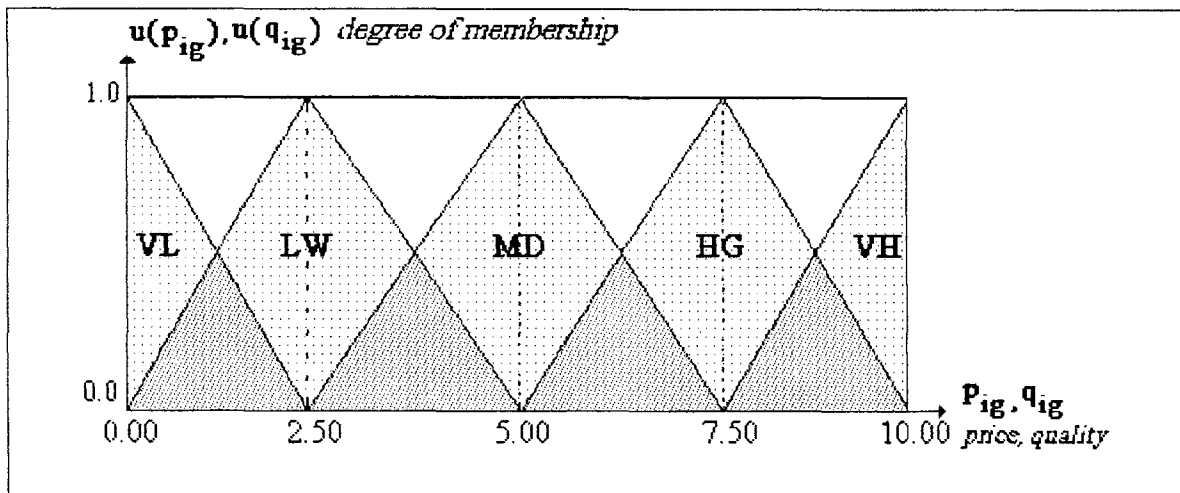


Figure 6.6 Basic Fuzzy Sets describing the Price p_{ig} and the Quality q_{ig} .

The equations for the Basic Fuzzy Sets in Figure 6.6 are not of interest because the standard configuration is only a starting point for designing the conclusive modified fuzzy sets that will represent the sensitivity parameters. Instead, the equations that determine the degree of membership of p_{ig} in the adjusted group of fuzzy sets are given. Two cases must be considered: $s_{pj} \geq 0$ or $s_{qj} \leq 0$ and $s_{pj} < 0$ or $s_{qj} > 0$. The equations for each situation regarding the price are 6.13(a) to 6.13(e) and 6.14(a) to 6.14(e), respectively. The same formulas apply to the quality, substituting $|s_{qj}|$ for s_{pj} and q_{ig} for p_{ig} .

$$s_{pj} \geq 0 \text{ or } s_{qj} \leq 0$$

$$u_{VL}(p_{ig}) = \begin{cases} 1 & \text{if } p_{ig} \leq s_{pj} \\ \left[0.4(s_{pj} - p_{ig}) + 1\right]^{\delta_j} & \text{if } s_{pj} < p_{ig} < s_{pj} + 2.5 \\ 0 & \text{otherwise} \end{cases} \quad \text{Eq. 6.13(a)}$$

$$u_{LW}(p_{ig}) = \begin{cases} \left[0.4(p_{ig} - s_{pj})\right]^{\delta_j} & \text{if } s_{pj} < p_{ig} \leq s_{pj} + 2.5 \\ \left[0.4(s_{pj} - p_{ig}) + 2\right]^{\delta_j} & \text{if } s_{pj} + 2.5 < p_{ig} < s_{pj} + 5.0 \\ 0 & \text{otherwise} \end{cases} \quad \text{Eq. 6.13(b)}$$

$$u_{MD}(p_{ig}) = \begin{cases} \left[0.4(p_{ig} - s_{pj}) - 1\right]^{\delta_j} & \text{if } s_{pj} + 2.5 < p_{ig} \leq s_{pj} + 5.0 \\ \left[0.4(s_{pj} - p_{ig}) + 3\right]^{\delta_j} & \text{if } s_{pj} + 5.0 < p_{ig} < s_{pj} + 7.5 \\ 0 & \text{otherwise} \end{cases} \quad \text{Eq. 6.13(c)}$$

$$u_{HG}(p_{ig}) = \begin{cases} \left[0.4(p_{ig} - s_{pj}) - 2\right]^{\delta_j} & \text{if } s_{pj} + 5.0 < p_{ig} \leq s_{pj} + 7.5 \\ \left[0.4(s_{pj} - p_{ig}) + 4\right]^{\delta_j} & \text{if } s_{pj} + 7.5 < p_{ig} \leq 10 \\ 0 & \text{otherwise} \end{cases} \quad \text{Eq. 6.13(d)}$$

$$u_{VH}(p_{ig}) = \begin{cases} \left[0.4(p_{ig} - s_{pj}) - 3\right]^{\delta_j} & \text{if } s_{pj} + 7.5 < p_{ig} \leq 10 \\ 0 & \text{otherwise} \end{cases} \quad \text{Eq. 6.13(e)}$$

$$s_{pj} < 0 \text{ or } s_{qj} > 0$$

$$u_{VL}(p_{ig}) = \begin{cases} [0.4(s_{pj} - p_{ig}) + 1]^{\lambda_j} & \text{if } 0 < p_{ig} < s_{pj} + 2.5 \\ 0 & \text{otherwise} \end{cases} \quad \text{Eq. 6.14(a)}$$

$$u_{LW}(p_{ig}) = \begin{cases} [0.4(p_{ig} - s_{pj})]^{\lambda_j} & \text{if } 0 < p_{ig} \leq s_{pj} + 2.5 \\ [0.4(s_{pj} - p_{ig}) + 2]^{\lambda_j} & \text{if } s_{pj} + 2.5 < p_{ig} < s_{pj} + 5.0 \\ 0 & \text{otherwise} \end{cases} \quad \text{Eq. 6.14(b)}$$

$$u_{MD}(p_{ig}) = \begin{cases} [0.4(p_{ig} - s_{pj}) - 1]^{\lambda_j} & \text{if } s_{pj} + 2.5 < p_{ig} \leq s_{pj} + 5.0 \\ [0.4(s_{pj} - p_{ig}) + 3]^{\delta_j} & \text{if } s_{pj} + 5.0 < p_{ig} < s_{pj} + 7.5 \\ 0 & \text{otherwise} \end{cases} \quad \text{Eq. 6.14(c)}$$

$$u_{HG}(p_{ig}) = \begin{cases} [0.4(p_{ig} - s_{pj}) - 2]^{\delta_j} & \text{if } s_{pj} + 5.0 < p_{ig} \leq s_{pj} + 7.5 \\ [0.4(s_{pj} - p_{ig}) + 4]^{\delta_j} & \text{if } s_{pj} + 7.5 < p_{ig} \leq 10 + s_{pj} \\ 0 & \text{otherwise} \end{cases} \quad \text{Eq. 6.14(d)}$$

$$u_{VH}(p_{ig}) = \begin{cases} [0.4(p_{ig} - s_{pj}) - 3]^{\delta_j} & \text{if } s_{pj} + 7.5 < p_{ig} \leq s_{pj} + 10 \\ 1 & \text{if } s_{pj} + 10 < p_{ig} \\ 0 & \text{otherwise} \end{cases} \quad \text{Eq. 6.14(e)}$$

For clarity, an example of the application of the method proposed to modify the Consumers' perceptions is given.

Example:

Suppose a Firm i and a Consumer j , both picked at random from their respective frequency distributions, are paired for an iteration. Let us assume that $p_{ig} = 6.78$, $c_{ig} = 5.67$

and $w_j = 0.456$. The first step is the qualification of the Normalized Income w_j employing the respective group of fuzzy sets (fuzzification) - Equations 6.10(a...e).

Income

$$u_{VL}(w_j = 0.456) = u_{LW}(0.456) = u_{VH}(0.456) = 0$$

$$u_{MD}(w_j = 0.456) = 0.468773$$

$$u_{HG}(w_j = 0.456) = 0.531227$$

Applying the rules of Table 6.5, comes:

Sensitivity to Price

$$u_{VS}(s_{pj}) = u_{ST}(s_{pj}) = u_{VI}(s_{pj}) = 0$$

$$u_{MS}(s_{pj}) = 0.468773$$

$$u_{IS}(s_{pj}) = 0.531227$$

and

Sensitivity to Quality

$$u_{VS}(s_{qj}) = u_{ST}(s_{qj}) = u_{VI}(s_{qj}) = 0$$

$$u_{MS}(s_{pj}) = 0.468773$$

$$u_{ST}(s_{pj}) = 0.531227$$

To perform the defuzzification, the method used is the superimposing *center of gravity of the areas*¹⁶ [KOSK92], [KOSK93], [VIOT93b]. The procedure employed here takes the areas of the fuzzy sets activated in the consequents of the production rules separately, therefore counting twice the overlapping regions to compute the centroid. The calculations yield the centroids $s_{pj} = 1.3021633$ and $s_{qj} = -1.3021633$, representing the

¹⁶ That method simplifies the arithmetic operations made by the simulation program. The elected option was found adequate, considering the great number of iterations to be performed during the simulations and, given the investigative character of the research, the lack of empirical data to check the sensitivity of the defuzzification method.

Consumer's sensitivity parameters for price and quality, respectively. Figure 6.7 illustrates the defuzzification process regarding s_{pj} . As to s_{qj} , the method is analogous.

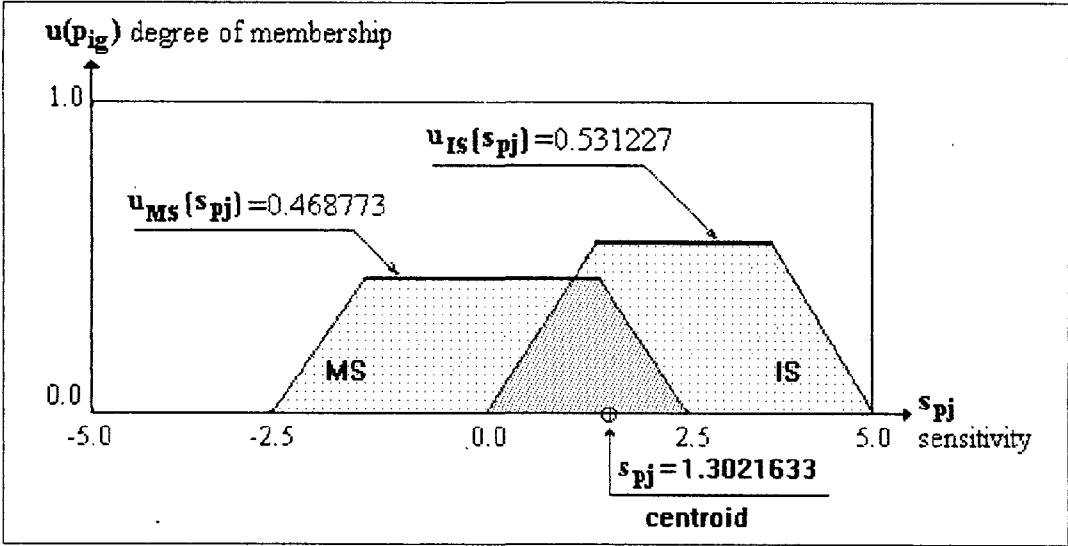


Figure 6.7 - Example of the Defuzzification of the Sensitivity to Price Parameter

Next, it is necessary to determine q_{ig} from c_{ig} using Equation 6.6, which yields $q_{ig}=7.447$.

The parameters s_{pj} and s_{qj} will alter the basic fuzzy sets which describe the perceptions of the price p_{ig} and the quality q_{ig} , according to the guidelines of Table 6.6. The universe of discourse for those variables remain unchanged in $[0, 10]$. The modified group of fuzzy sets for the price are illustrated in Figure 6.8, with a similar procedure is performed for the quality.

$$s_{pj} = 1.3021633; s_{qj} = -1.3021633$$

$$\text{Dilation operator (Eq. 6.12(a))}: \delta_j = \frac{1}{1 + \frac{1.3021633}{5}} = 0.7934$$

$$\text{Concentration operator (Eq. 6.12(b))}: \chi_j = 1 + \frac{1.3021633}{5} = 1.2604$$

Then, to the Consumer chosen for the example, whose income is $w_j=0.456$, the price and the quality of the product or service in perspective offered by Firm i shall be qualified by the respective modified fuzzy sets, using equations 6.13(a...e) and 6.14(a...e).

Price

$$u_{VL}(p_{ig}=6.78)=u_{LW}(p_{ig}=6.78)=u_{VH}(p_{ig}=6.78)=0; \text{ (Eqs. 13(a,b,e))}$$

$$u_{MD}(p_{ig}=6.78) = 0.7654 ; \text{ (Eq. 6.13(c))}$$

$$u_{HG}(p_{ig}=6.78) = 0.1242 ; \text{ (Eq. 6.13(d))}$$

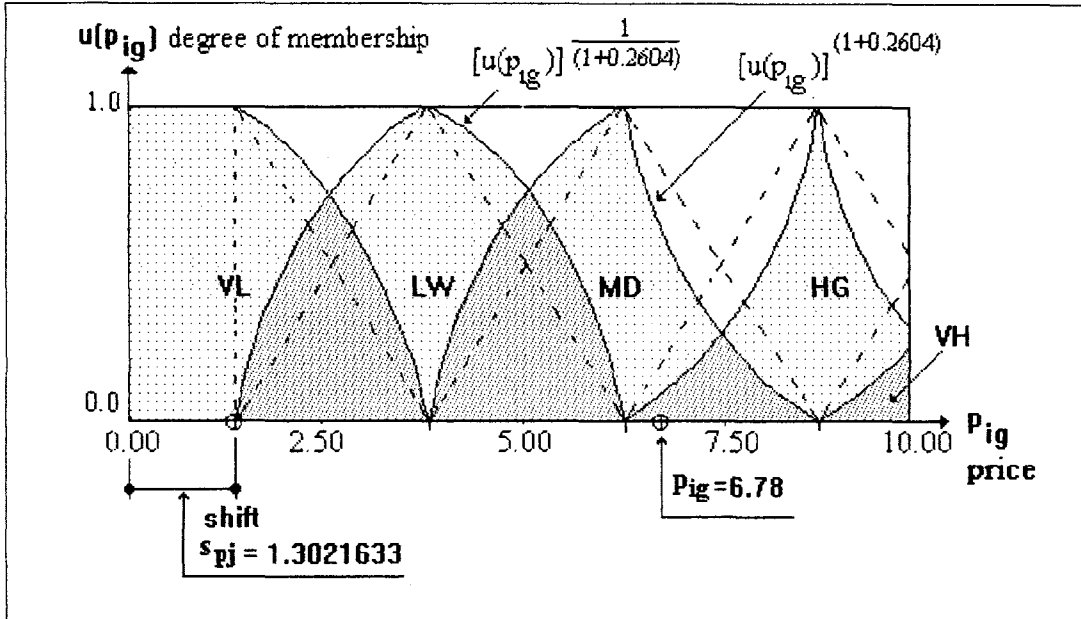


Figure 6.8 Modified Fuzzy Sets according to a Consumer's Sensitivity to Price

Quality

$$u_{VL}(q_{ig}=7.447)=u_{LW}(q_{ig}=7.447)=u_{VH}(q_{ig}=7.447)=0; \text{ (Eqs. 6.13(a,b,e))}$$

$$u_{MD}(q_{ig}=7.447) = 0.4622 ; \text{ (Eq. 6.13(c))}$$

$$u_{HG}(q_{ig}=7.447) = 0.3737 ; \text{ (Eq. 6.13(d))}$$

of its subjective evaluation of the deal. The criteria and mechanism employed to perform that evaluation are explained in the following section.

Let us suppose that a Consumer has decided to buy an item from a particular supplier (Firm)¹⁷. After the purchase has been consummated, the buyer observes the quality of the merchandise, and with its own perceptions achieved for the factors *price* and *quality*, it will then perform an evaluation of the deal. This appraisal will serve as the basis for the assignment of payoffs to this category of players.

In order to carry out this objective, each of the perceptions of the two mentioned factors will again constitute the inputs of a fuzzy expert system, each of which yields as the output a new variable, to be called the *attractiveness* α_{pj} of the price p_{ig} , and of the quality α_{qj} as perceived by the Consumer j .

The variables α_{pj} and α_{qj} , likewise other former parameters present in the game's model, shall be described qualitatively by fuzzy sets. The universe of discourse is the interval $[0, 1]$, which is divided into similar fuzzy sets for both attributes α_{pj} and α_{qj} . The mapping of p_{ig} into α_{pj} and q_{ig} into α_{qj} will be made with the aid of the fuzzy production rules listed in Table 6.8 and 6.9, respectively.

Figures 6.10 and 6.11 depict the fuzzy sets qualifying the attractiveness of the price p_{ig} and of the quality q_{ig} , both modulated according to every Consumer's sensitivity parameters s_{pj} and s_{qj} .

¹⁷ The explanation of how the Consumer has arrived at that decision is deferred to a subsequent section of this work. See 6.4.7.

| | |
|----------------|--|
| Rule 1: | <i>If the price p_{ig} is Very High (VH) then the attractiveness of the price α_{pj} is Very Unattractive (VU).</i> |
| Rule 2: | <i>If the price p_{ig} is High (HG) then the attractiveness of the price α_{pj} is Unattractive (NA).</i> |
| Rule 3: | <i>If the price p_{ig} is Medium (MD) then the attractiveness of the price α_{pj} is Neutral (NT).</i> |
| Rule 4: | <i>If the price p_{ig} is Low (LW) then the attractiveness of the price α_{pj} is Attractive (AT).</i> |
| Rule 5: | <i>If the price p_{ig} is Very Low (VL) then the attractiveness of the price α_{pj} is Very Attractive (VA).</i> |

Table 6.8 - Rules used by the Fuzzy Expert System that associates the Price p_{ig} as perceived by a Consumer to its correspondent Attractiveness

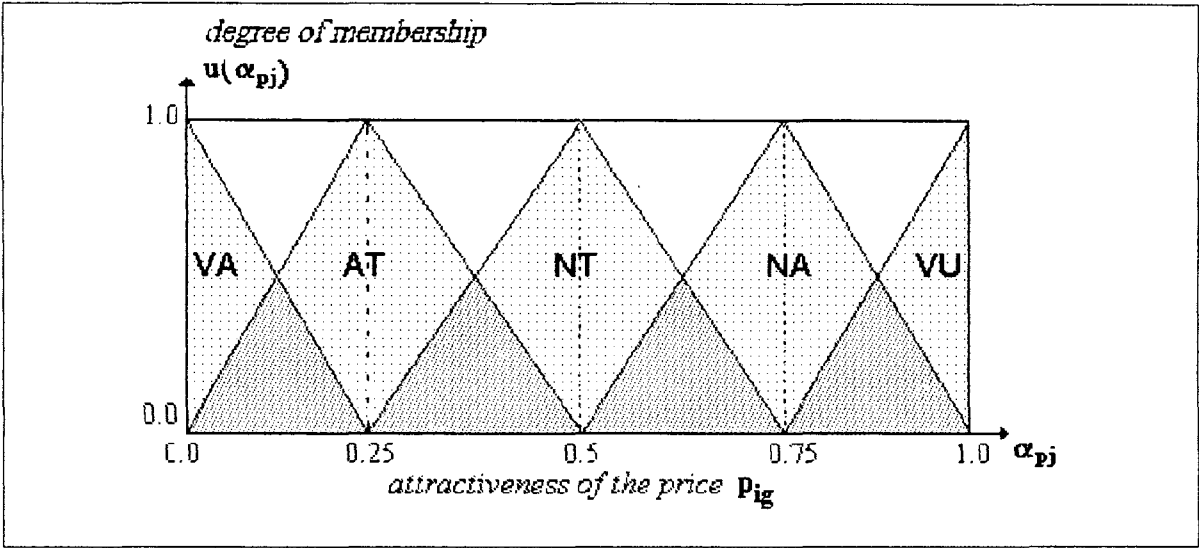


Figure 6.10 - Fuzzy Sets depicting the Attractiveness of the Price p_{ig}

- Rule 1:** *If the quality q_{ig} is Very High (VH) then the attractiveness of the quality α_{qj} is Very Attractive (VA).*
- Rule 2:** *If the quality q_{ig} is High (HG) then the attractiveness of the quality α_{qj} is Attractive (AT).*
- Rule 3:** *If the quality q_{ig} is Medium (MD) then the attractiveness of the quality α_{qj} is Neutral (NT).*
- Rule 4:** *If the quality q_{ig} is Low (LW) then the attractiveness of the quality α_{qj} is Unattractive (NA).*
- Rule 5:** *If the quality q_{ig} is Very Low (VL) then the attractiveness of the quality α_{qj} is Very Unattractive (VU).*

Table 6.9 - Rules used by the Fuzzy Expert System that associates the Quality q_{ig} as perceived by a Consumer to its correspondent Attractiveness

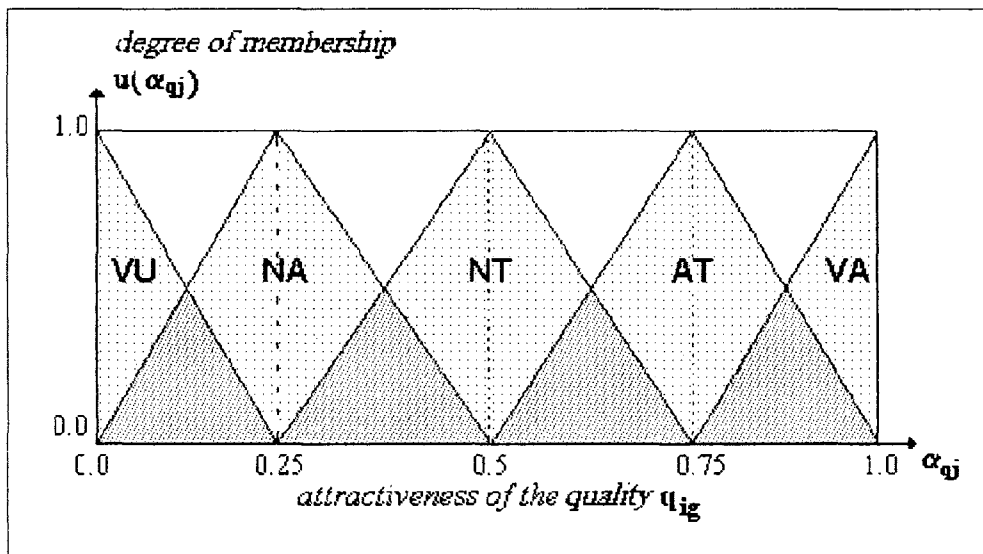


Figure 6.11 - Fuzzy Sets depicting the Attractiveness of the Quality q_{ig}

Applying the fuzzy production rules of Tables 6.8 and 6.9 and defuzzifying the results obtained from the respective group of fuzzy sets, the method will generate one

specific value for each of the variables α_{pj} and α_{qj} . It is rather intuitive that the payoff to be assigned to a player should reflect, as well as possible, the *utility*, or level of satisfaction that is obtained from the product or service purchased. It can also be claimed that the evaluation of the item by means of the attractiveness factors may be considered a reasonable approach to that problem. So, the payoff which results from an accomplished transaction depends on a trade-off between the attributes price and quality. The problem consists in the assessment, by an individual referee (the Consumer), of the gain obtained with the purchase of the item (object). In the present case, the object has an attribute space $X=\{price, quality\}$.

One traditional approach to that evaluation problem is the method of the weighted mean. Although still largely employed, this technique relies on the presupposition that the attributes of the object (attractiveness of price, quality) are independent of one another, and that their individually assigned scores may be added to provide a final conclusion. Nevertheless, as pointed out by Wang and Klir [WANG92], “()... in most real problems, these effects are interactive...; If we adopt a nonadditive set function (a fuzzy measure) to characterize the importance of the two factors and, relevantly, use **Fuzzy Integrals** as a synthetic evaluator of the quality ... a satisfactory result may be obtained.”

Following the procedure proposed by Wang and Klir, each subset *price, quality* of the attribute space X is connected with a real number $\phi(price)$, $\phi(quality)$, $\phi(.) \in [0,1]$, that will stand for the importance¹⁸ of each respective attribute.

Given any generic subsets of the attribute space X , e.g E and F , the function $\phi(.)$ must obey the conditions:

- $\phi(\emptyset) = 0$ and $\phi(X) = 1$;
- If $E \subset F \subset X$, then $\phi(E) \leq \phi(F)$

The sole attributes of the product or service being judged are price and quality, and they do not have sub attributes. So, only one space exists in this model.

¹⁸ The set function corresponds to the *weights* in the weighted mean method. However, $\phi(.)$ does not have the additive property, so, for instance, $\phi(x_1) + \phi(x_2) \neq \phi(x_1 + x_2)$. See Op. Cit. [WANG92].

An important characteristic of Wang and Klir's method is that the sum of the relevance measures of any attributes taken alone is not equal to that of the combination. In other words, the object's properties reinforce each other.

The fuzzy integral $\int_F \alpha \partial \phi$, where α is the attractiveness of the object (product or service) being evaluated, may be employed for the aggregation of the attributes price p_{ig} and quality q_{ig} regarding their respective importance mirrored by the set function $\phi(\cdot)$.

To compute $\phi(\text{price})$ and $\phi(\text{quality})$ once again the Consumers' respective Incomes will be used. This criterion has been preferred rather than an arbitrary setting of constant numeric values because it allows the individualization of every evaluator (Consumer). Therefore, each Consumer is made unique by its sensitivities to price and quality (s_{pj} , s_{qj}) and also by its importance factors $\phi(\text{price})$ and $\phi(\text{quality})$, which are expressed as $\phi(s_{pj})$ and $\phi(s_{qj})$. The importance factors have been designed to lie in the interval $[0.1, 0.4]$, and their sum is not constant, since the idea derived from the proposed use of fuzzy integrals is that the importance of the attributes are mutually strengthened if they are contemplated conjointly.

In order to implement the assignment of values to $\phi(s_{pj})$ and $\phi(s_{qj})$, they shall be defined as an increasing functions of $(w_{\max} - w)$ and $(w - w_{\min})$ respectively, as shown in Equations 6.15 and 6.16.

$$\phi(s_{pj}) = \frac{(w_{\max} - w) \times 0.3}{w_{\max}} + 0.1 \quad \text{Eq. 6.15(a)}$$

$$\text{and} \quad \phi(s_{qj}) = \frac{(w - w_{\min}) \times 0.3}{w_{\max}} + 0.1 \quad \text{Eq. 6.15(b)}$$

Figure 6.12 shows the curves illustrating the behavior of the functions.

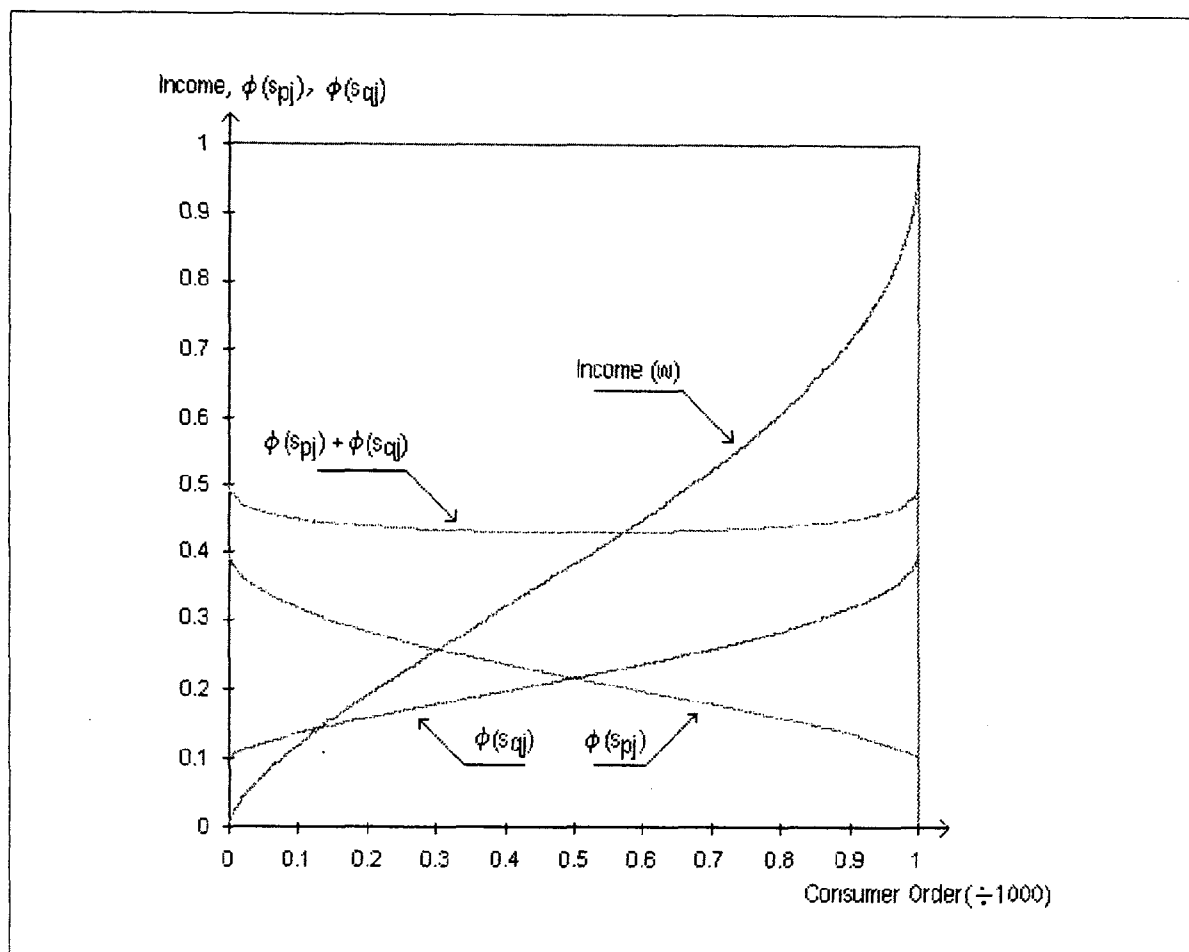


Figure 6.12 The Importance Factors of the Attributes Price , Quality.

The combined importance of the two attributes s_{pj} and s_{qj} taken conjointly is 1, which refers to the importance of the whole attribute space X , that is, $\phi(X) = \phi(s_{pj} + s_{qj}) = 1$. Following this procedure, an item shall have its *basic synthetic evaluation* σ_{igj} ' given by Equation 6. 16.

$$\sigma_{igj}^{\circ} = \int_F \alpha \partial \phi = [\min(\alpha_{pj}, \alpha_{qj}) \wedge \phi(s_{pj} + s_{qj})] \vee [\max(\alpha_{pj}, \alpha_{qj}) \wedge \phi(s_{pj} \text{ or }^{19} s_{qj})] \quad \text{Eq. 6.16}$$

¹⁹ The argument of $\phi(\cdot)$ to be used in the second fuzzy intersection operation inside the basic fuzzy integral is the attractiveness index α with the subscript **pj** or **qj** which is **different** from the one that resulted from the previous **min** operation.

Taking into account the discussion contained in Chapter 4 of this Dissertation, the *final synthetic evaluation* σ_{igl} of the object with the basic fuzzy integral $\sigma^{\circ}_{\text{igl}} = \int_{\text{F}} \alpha \partial \phi$ of the attractiveness indexes α_{pj} and α_{qj} with respect to their importance measures $\phi(s_{\text{pj}})$ and $\phi(s_{\text{qj}})$ will be modified from its original formulation. According to the alteration proposed, the *final synthetic evaluation* σ_{igl} will then consist of a sum of three values: (1) the original fuzzy integral (Eq. 6.16); (2) an increment (Δ^+), which corresponds to the gain obtained by an eventual better score of an attribute (price or quality) that is not affecting the original fuzzy integral, and (3) a decrement (Δ^-), relative to a loss derived from an prospective worse feature that is not critical on the value yielded by the original fuzzy integral. The increment has been taken proportional to $\phi(\cdot)$ and decrement proportional to $[\phi(\cdot)]^{\frac{1}{2}}$. The reason for this approach regards a presupposed greater concern to losses than to gains. As both $\phi(s_{\text{pj}})$ and $\phi(s_{\text{qj}})$ mirror the importance a particular Consumer assigns to the price and quality of an item, the influence exerted on (Δ^+) and (Δ^-) will be compatible.

| Condition | Increment (Δ^+) & Decrement (Δ^-) | |
|--|---|-------------|
| $\alpha_{\text{pj}} \leq \alpha_{\text{qj}}$ | $\Delta^+ = \phi(s_{\text{qj}}) \times \{\max [0, (\alpha_{\text{qj}} - \max (\alpha_{\text{pj}}, \max(\alpha^{\circ}_{\text{qj}}, \phi(s_{\text{qj}}))))]\}$ | Eq. 6.17(a) |
| $\alpha_{\text{pj}} > \alpha_{\text{qj}}$ | $\Delta^+ = \phi(s_{\text{pj}}) \times \{\max [0, (\alpha_{\text{pj}} - \max (\alpha^{\star}_{\text{pj}}, \max((\alpha_{\text{qj}}, \phi(s_{\text{pj}})))))]\}$ | Eq. 6.17(b) |
| $\alpha_{\text{pj}} \leq \alpha_{\text{qj}}$ | $\Delta^- = \phi(s_{\text{pj}})^{\frac{1}{2}} \times \{\min [0, (\alpha_{\text{pj}} - \min (\phi(s_{\text{qj}}), \alpha_{\text{qj}}))]\}$ | Eq. 6.17(c) |
| $\alpha_{\text{pj}} > \alpha_{\text{qj}}$ | $\Delta^- = \phi(s_{\text{qj}})^{\frac{1}{2}} \times \{\min [0, (\alpha_{\text{qj}} - \min (\phi(s_{\text{pj}}), \alpha_{\text{pj}}))]\}$ | Eq. 6.17(d) |

Then, the final synthetic evaluation is depicted by Equation 6.18.

$$\sigma_{igj} = \sigma^{\circ}_{igj} + \Delta^{+} + \Delta^{-} \quad \text{Eq. 6.18}$$

The values of σ_{igj} will be mapped into the final payoffs, after a scale transformation that equals the minimum and maximum attainable values of σ_{igj} to -1 and 1, respectively. The determination of the Consumer's payoff z_{igj} ²⁰ in a given iteration y , $y = 1, 2, \dots, b_{jlg}$ with Firm i in the cycle g if (and only if) it purchased the product or service is provided is explained in the next section.

6.4.7 - The Consumer's Payoffs

Once the item has been evaluated by means of the transformed fuzzy integral, being ascribed a value of σ_{igj} , a payoff must be assigned to the buyer, reflecting its personal gain (or loss) with the effectuated transaction. However, the link between the product or service's synthetic evaluation σ_{igj} and the correspondent Consumer's payoff z_{igj} is not straightforward, so a discussion of this matter is needed in order to arrive at an adequate scale for z_{igj} . For this purpose, some considerations about the *range* of σ_{igj} are necessary.

First, it is important to recall that the attractiveness indexes α_{pj} and α_{qj} are obtained through the application of a group of fuzzy production rules regarding an item's price and quality. Since the characterizations of price and quality depend on the Consumer's sensitivities s_{pj} and s_{qj} , the mappings $p_{ig} \rightarrow \alpha_{pj}$ and $q_{ig} \rightarrow \alpha_{qj}$ will be different for each particular buyer j . The importance measures $\phi(s_{pj})$ and $\phi(s_{qj})$ are also individualized per Consumer.

²⁰ The assignment of the three subscripts igj to the synthetic evaluation σ and the payoff z was necessary in order identify the particular pair of players (i, j) as well as the cycle g to which σ and z refer. Still, to preserve the history of the iterations' results to be generated in the simulations, a fourth superscript y regarding the iterations' sequence order will be added to the payoff z . On the other hand, the attractiveness indexes α_{pj} , α_{qj} do not need those expedients, since they are used only momentarily, and can be dynamically erased and updated by the program during runtime.

Second, the basic fuzzy integral depicted in Equation 6.16 can yield only four possible results, depending on the values of α_{pj} , α_{qj} , $\phi(s_{pj})$ and $\phi(s_{qj})$. The feasible values of σ_{igj} are shown in Table 6.10, according to every workable hypothesis.

| Hypothesis | | $\int_F \alpha \partial \phi$ | σ_{igj}^o |
|-----------------------------|--|---|------------------|
| $\alpha_{qj} > \alpha_{pj}$ | $\phi(s_{qj}) > \alpha_{qj} > \alpha_{pj}$ | $(\alpha_{pj} \wedge 1) \vee (\alpha_{qj} \wedge \phi(s_{qj}))$ | α_{qj} |
| | $\alpha_{qj} > \phi(s_{qj}) > \alpha_{pj}$ | $(\alpha_{pj} \wedge 1) \vee (\alpha_{qj} \wedge \phi(s_{qj}))$ | $\phi(s_{qj})$ |
| | $\alpha_{qj} > \alpha_{pj} > \phi(s_{qj})$ | $(\alpha_{pj} \wedge 1) \vee (\alpha_{qj} \wedge \phi(s_{qj}))$ | α_{pj} |
| $\alpha_{qj} < \alpha_{pj}$ | $\phi(s_{pj}) < \alpha_{qj} < \alpha_{pj}$ | $(\alpha_{qj} \wedge 1) \vee (\alpha_{pj} \wedge \phi(s_{pj}))$ | α_{qj} |
| | $\alpha_{qj} < \phi(s_{pj}) < \alpha_{pj}$ | $(\alpha_{qj} \wedge 1) \vee (\alpha_{pj} \wedge \phi(s_{pj}))$ | $\phi(s_{pj})$ |
| | $\alpha_{qj} < \alpha_{pj} < \phi(s_{pj})$ | $(\alpha_{qj} \wedge 1) \vee (\alpha_{pj} \wedge \phi(s_{pj}))$ | α_{pj} |

Table 6.10 - Possible Values of the Basic Synthetic Evaluation σ_{igj}^o

While $\phi(s_{pj})$, $\phi(s_{qj})$ are parameters not related to a product or service's attributes and depend only on each Consumer's Income, the variables α_{qj} , α_{pj} , besides influenced by s_{pj} and s_{pj} , are obviously also a function of an item's correspondent characteristics.

As an illustration of the process of determination of σ_{igj} , a sample of the curves showing the variables and parameters involved is shown in Figure 6.13. The example illustrates the situation relative to the Consumer with $s_{pj} = 0$, which corresponds to the *median* Income of the population, henceforth to be called *typical*. In the graph, the point where $\alpha_{pj} = \alpha_{qj}$ coincides with the maximum score²¹, regarding the synthetic evaluation σ_{igj} , that can be assigned by this specific buyer to any product or service.

²¹ This situation does not necessarily occur for every value of s_{pj} .

Note that, according to the rules of the game, a supplier can set any cost to the item it is currently selling, only provided that it is less or at most equal to its price.

The curves showing the behavior of α_{pj} and α_{qj} for other sensitivities values are included in the Appendix²².

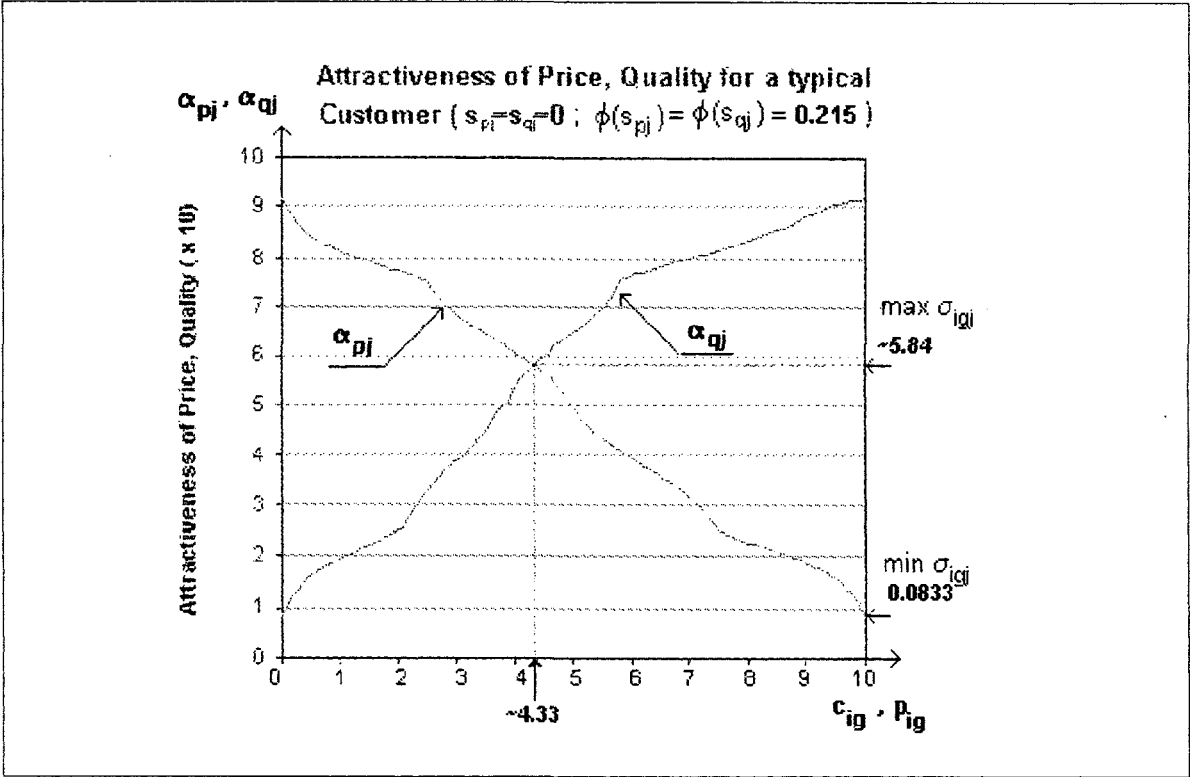


Figure 6.13 - Attractiveness Indexes α_{pj} , α_{qj} , and the Range of the Synthetic Evaluation σ_{igj}

As can be seen from Figure 6.13, the behavior of α_{pj} , α_{qj} is far from a smooth curve, even in the simplest case as displayed, of the *typical* Consumer. The same can be said of σ_{igj} .

The maximum and minimum values of σ_{igj} that can be assigned to any item for every combination of price and quality are listed in Table 6.11. They were determined

²² The curves were obtained employing the fuzzy expert system (Tables 6.8, 6.9) regarding the qualification of price and quality and defuzzifying the corresponding attractiveness (Figures 6.10 and 6.11). The calculations were performed using a spreadsheet.

applying Equation 6.18 for eleven points regarding s_{pj} . Figure 6.14 illustrates the corresponding graph.

| | Sensitivities $s_{pj} = -s_{cj}$ | | | | | | | | | | |
|-------------------|----------------------------------|------|------|------|------|------|------|------|------|------|------|
| | -5 | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 | 5 |
| MIN σ_{ij} | .089 | .106 | .115 | .132 | .137 | .083 | .137 | .132 | .109 | .106 | .089 |
| MAX σ_{ij} | .500 | .509 | .509 | .548 | .572 | .583 | .597 | .603 | .598 | .561 | .500 |

Table 6.11 - Maximum and Minimum Values for the Synthetic Evaluation of an Item

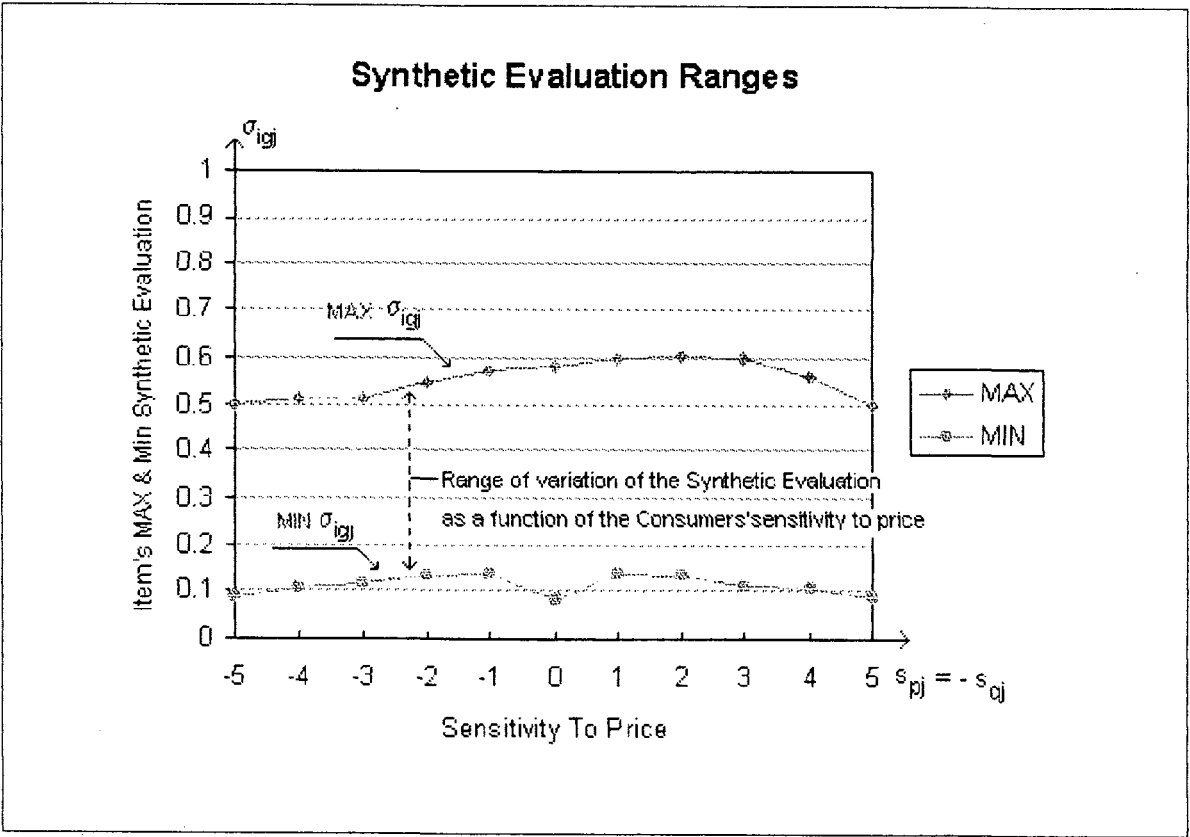


Figure 6.14 - Ranges of the Synthetic Evaluation as a Function of the Consumers' Sensitivities to Price

A convenient range for the payoffs is the interval $[-1,1]$. In order to associate the values of the synthetic evaluation to that scale, the *minimum* and *maximum* values that may be assumed by σ_{igj} for each s_{pj} shall correspond the extreme points -1 and 1, respectively. In this manner, each Consumer will have an individual scale, and consequently, a specific linear transformation to perform the mapping $\sigma_{igj} \rightarrow z_{igj}$. The quantities $\text{MAX } \sigma_{igj}$ and $\text{MIN } \sigma_{igj}$ are non-linear functions of (s_{pj}, s_{qj}) , but to avoid unessential computational burden, these values will be inferred from linear regressions for ten intervals of variation of the sensitivity parameters $s_{pj} = -s_{qj}$. Table 6.12 lists the respective regression equations, calculated from the data contained in Table 6.11 and Figure 6.14.

| Range of s_{pj} | $\text{MIN } \sigma_{igj} = f(s_{pj})$ | $\text{MAX } \sigma_{igj} = f(s_{pj})$ |
|-------------------|--|--|
| $[-5, -4)$ | $0.171379 + 0.016304 s_{pj}$ | $0.547475 + 0.09495 s_{pj}$ |
| $[-4, -3)$ | $0.142639 + 0.009199 s_{pj}$ | $0.507731 - 0.000441 s_{pj}$ |
| $[-3, -2)$ | $0.165835 + 0.016931 s_{pj}$ | $0.625640 + 0.038862 s_{pj}$ |
| $[-2, -1)$ | $0.141339 + 0.004683 s_{pj}$ | $0.595290 + 0.023687 s_{pj}$ |
| $[-1, 0)$ | $0.083333 - 0.053323 s_{pj}$ | $0.581965 + 0.10362 s_{pj}$ |
| $[0, 1)$ | $0.083333 + 0.053248 s_{pj}$ | $0.581965 + 0.015030 s_{pj}$ |
| $[1, 2)$ | $0.141320 - 0.004739 s_{pj}$ | $0.591281 + 0.005714 s_{pj}$ |
| $[2, 3)$ | $0.177014 - 0.022586 s_{pj}$ | $0.611647 - 0.004469 s_{pj}$ |
| $[3, 4)$ | $0.120173 - 0.003639 s_{pj}$ | $0.708553 - 0.036771 s_{pj}$ |
| $[4, 5)$ | $0.170257 - 0.016160 s_{pj}$ | $0.807345 - 0.061469 s_{pj}$ |

Table 6.12 - Regression Equations for the Extreme Values that may be assumed by σ_{igj} as a Function of s_{pj}

With the MIN and MAX values of σ_{igj} available, the mapping $\sigma_{igj} \rightarrow z_{igj}$, with $z_{igj} \in [-1, 1]$ is forthright, using the general linear transformation of Equation 6.19.

$$z_{igj} = 2 \times \frac{\sigma_{igj} - \text{MIN}\sigma_{igj}}{\text{MAX}\sigma_{igj} - \text{MIN}\sigma_{igj}} - 1 \quad \text{Eq. 6.19}$$

Table 6.13 presents the specific equations for each range of s_{pj} .

| Range of s_{pj} | $z_{igj} = f(s_{pj}, \sigma_{igj})$ | Range of s_{pj} | $z_{igj} = f(s_{pj}, \sigma_{igj})$ |
|-------------------|---|-------------------|---|
| [-5, -4) | $z_{igj} = \frac{\sigma_{igj} - 0.171379 - 0.016384 s_{pj}}{0.188048 - 0.00344 s_{pj}} - 1$ | [0, 1) | $z_{igj} = \frac{\sigma_{igj} - 0.08333 - 0.053248 s_{pj}}{0.249316 - 0.019115 s_{pj}} - 1$ |
| [-4, -3) | $z_{igj} = \frac{\sigma_{igj} - 0.142639 - 0.009199 s_{pj}}{0.182546 - 0.004825 s_{pj}} - 1$ | [1, 2) | $z_{igj} = \frac{\sigma_{igj} - 0.14132 + 0.04739 s_{pj}}{0.224981 + 0.005227 s_{pj}} - 1$ |
| [-3, -2) | $z_{igj} = \frac{\sigma_{igj} - 0.165835 - 0.016931 s_{pj}}{-0.229903 - 0.010966 s_{pj}} - 1$ | [2, 3) | $z_{igj} = \frac{\sigma_{igj} - 0.177014 + 0.02586 s_{pj}}{0.217317 + 0.09059 s_{pj}} - 1$ |
| [-2, -1) | $z_{igj} = \frac{\sigma_{igj} - 0.141339 - 0.04683 s_{pj}}{0.226976 + 0.09502 s_{pj}} - 1$ | [3, 4) | $z_{igj} = \frac{\sigma_{igj} - 0.120173 + 0.003639 s_{pj}}{0.29419 - 0.01657 s_{pj}} - 1$ |
| [-1, 0) | $z_{igj} = \frac{\sigma_{igj} - 0.08333 + 0.05332 s_{pj}}{0.249316 + 0.031843 s_{pj}} - 1$ | [4, 5] | $z_{igj} = \frac{\sigma_{igj} - 0.170257 + 0.1616 s_{pj}}{0.3185444 - 0.02265 s_{pj}} - 1$ |

Table 6.13 - Consumers' Payoffs as Linear Functions of σ_{igj} and s_{pj}

As the Consumers' payoffs z_{igj} belong to the interval $[-1, 1]$, the midpoint zero means indifference, neither a gain nor a loss.

It should be observed that the Firms’ payoffs, already established in Section 6.4.4-I, belong to a quite different interval, namely $[-(t_1 + t_2), 10)^{23}$. As a matter of fact, the distinct intervals chosen to contain the feasible payoffs for each category of players (buyers or sellers) do not have to be alike. In this model, the problem of comparison of utilities between the two categories of players is not relevant, so the scales with which the payoffs are measured can be independent for Firms and Consumers. However, the cardinal valuation of the gains have a significant role in the decision making of each class of contenders. The Consumers will take their payoffs into account to arrive at the decision of whether to purchase or not an item, and for the Firms, the gains accumulated in the course of the iterations are the measure of the degree of success they are reaching with their policies of cost, price and advertising. The game’s payoffs are summarized in Table 6.14.

| Player's Payoffs | Consumer's Decisions | |
|--------------------------------------|---------------------------|--|
| | Buys | Does not buy |
| Firm: $(r_{ig} \leq 10)$ | $p_{ig} - c_{ig}$ | $-\left(t_1 + t_2 \times \frac{c_{ig}}{p_{ig}}\right)$ |
| Consumer: $(-1 \leq z_{igj} \leq 1)$ | $f(\sigma_{igj}, s_{pj})$ | 0 |

Table 6.14- Summary of the Game's Payoffs

In order to consolidate the concepts just presented, the example developed in section 6.4.5 will be extended incorporating the item’s evaluation and the determination of the Consumer’s payoff.

²³ The parameters t_1 and t_2 are intended to have the default values $t_1 = 0$ and $t_2 = 0.1$, but they can be differently arbitrated before the game commences, and remain constant throughout the simulations in a cycle; They represent a calibration of the penalty that is imposed to a Firm when its product or service is refused by a customer.

Example (continued from page 6.33)

Recalling the perceived characteristics of the item purchased:

Price

$$u_{VL}(p_{ig}=6.78)=u_{LW}(p_{ig}=6.78)=u_{VH}(p_{ig}=6.78)=0;$$

$$u_{MD}(p_{ig}=6.78) = 0.7654;$$

$$u_{HG}(p_{ig}=6.78) = 0.1242.$$

Quality

$$u_{VL}(q_{ig}=7.447)=u_{LW}(q_{ig}=7.447)=u_{VH}(q_{ig}=7.447)=0;$$

$$u_{MD}(q_{ig}=7.447) = 0.4622;$$

$$u_{HG}(q_{ig}=7.447) = 0.3737.$$

The next phase is the calculation of the attractiveness indexes for the item. Applying the rules from Table 6.8 and 6.9 in the fuzzy sets shown in Figures 6.10 and 6.11 respectively for α_{pj} and α_{qj} , comes:

Attractiveness of Price

$$u_{VA}(\alpha_{pj})=u_{AT}(\alpha_{pj})=u_{VT}(\alpha_{pj})=0;$$

$$u_{NT}(\alpha_{pj}) = 0.7654;$$

$$u_{NA}(\alpha_{pj}) = 0.1242.$$

Attractiveness of Quality

$$u_{VT}(\alpha_{qj})=u_{NA}(\alpha_{qj})=u_{VA}(\alpha_{qj})=0;$$

$$u_{NT}(\alpha_{qj}) = 0.4622;$$

$$u_{AT}(\alpha_{qj}) = 0.3737.$$

Defuzzifying α_{pj} and α_{qj} using the method of the center of gravity of area yields:

$$\alpha_{pj} = 0.4506 \text{ and}$$

$$\alpha_{qj} = 0.6152.$$

As mentioned before, α_{pj} and α_{qj} are the *scores* the Consumer assigns to the product regarding price and quality separately. To reach the final, or aggregated evaluation of the item, these two scores must be integrated (in a fuzzy manner) with respect to the importance factors $\phi(s_{pj})$ and $\phi(s_{qj})$.

The Consumer's sensitivity parameters are: $s_{pj}=1.302163$; $s_{qj}=-1.302163$; Referring to Equation 6.15, comes:

$$\phi(s_{pj}) = 0.192437,$$

$$\phi(s_{qj}) = 0.239039 \text{ and}$$

$$\phi(s_{pj} + s_{qj}) = 1.$$

Aggregating the scores and importance factors using the fuzzy integral of Equation 6.18, the *basic* fuzzy integral yields:

$$\begin{aligned} \sigma_{igj}^{\circ} &= \int_F \alpha \partial \phi = [\min(\alpha_{pj}, \alpha_{qj}) \wedge \phi(s_{pj} + s_{qj})] \vee [\max(\alpha_{pj}, \alpha_{qj}) \wedge \phi(s_{pj} \text{ or } s_{qj})] \\ &= \max\{\min(\min(0.4506, 0.6152), 1), \min[0.6152, 0.239039]\} \\ &= \mathbf{0.4506} \end{aligned}$$

The increment Δ^+ and decrement Δ^- are, for $(\alpha_{pj} \leq \alpha_{qj})$:

$$\begin{aligned} \Delta^+ &= \phi(s_{qj}) \times \{\max[0, (\alpha_{qj} - \max(\alpha_{pj}, \max(\alpha_{qj}^{\circ}, \phi(s_{qj}))))]\} \\ &= 0.239039 \times \{\max[0, (0.6152 - \max(0.4506, \max(0.08333, 0.239039)))]\} \\ &= \mathbf{0.03934582} \end{aligned}$$

$$\begin{aligned}
\Delta^- &= \phi(s_{pj})^{\frac{1}{2}} \times \{\min[0, (\alpha_{pj} - \min(\phi(s_{qj}), \alpha_{qj}))]\} \\
&= 0.192437^{\frac{1}{2}} \times \{\min[0, (0.4506 - \min(0.230939, 0.6152))]\} \\
&= 0
\end{aligned}$$

The final value for the synthetic evaluation will be:

$$\sigma_{igj} = \sigma_{igj}^o + \Delta^+ + \Delta^- = 0.4506 + 0.03934582 + 0 = 0.489946$$

According to Table 6.13, the payoff z_{igj} will be obtained through the equation

$$z_{igj} = \frac{\sigma_{igj} - 0.14132 + 0.004739s_{pj}}{0.224981 + 0.005227s_{pj}} - 1,$$

which corresponds to the range of $s_{pj}=1.3022$.

Then, substituting for $s_{pj}=1.3022$ and $\sigma_{igj}=0.489946$,

$$z_{igj}=0.368904$$

One important point to mention is that the Consumer's payoff which derives from a completed transaction is not directly related to the amount paid for the product or service purchased. This fact is quite comprehensible under the assumptions used in the model, which ascribe different *utilities* for the monetary unit, depending on the individual's sensitivity to price. In the current example, a payoff *zero*, besides being obtainable when the deal is not accomplished, could also be attainable by a product with a synthetic evaluation $\sigma_{igj} \cong 0.37249$ ²⁴. Considering that the price $p_{ig}=6.78$ did not change, the decrease in the synthetic evaluation is necessarily implied from a reduction in the quality of the item. Using the curves for α_{qj} from the Appendix and linearly interpolating, a cost $c_{ig} \cong 3.2$ would result for the item as the one that corresponds to $z_{igj}=0$. In other words, this

²⁴ Using the correspondent equation for $s_{pj}=1.302$ from Table 6.13.

particular Consumer is indifferent in buying for 6.78 MU an item which cost 3.2 MU to the supplier to produce, or not acquiring it at all.

6.4.8 - The Consumer's Decision

A crucial phase of the game is the Consumer's decision whether it will or will not consummate the transaction which is being offered to it by Firm i . To arrive at this important conclusion, when confronted with the mentioned dichotomic choice, a potential buyer j will basically take into account two inputs:

- a) An estimate of the payoff that it would obtain *if* it buys the product or service. Note that this appeasement is performed under *incomplete*, or *partial* information, because the buyer can have direct access to only one of the item's attributes, that is, the price p_{ig} ²⁵.
- b) A reference value, valid for this specific occasion, to which the Consumer will match its estimate. This reference has the role of a *threshold* for the Consumer's decision making.

The result that outcomes from the cited comparison will indicate whether the transaction is accomplished. The two mentioned inputs will be separately discussed in the sequence.

1. The payoff estimate

As demonstrated in section 6.4.7, the payoff z_{igj} is a linear and increasing function of σ_{igj} . This transformation of scale became necessary in order to establish a common reference measure of the gains /losses achieved in the iterations for the whole population of Consumers. Recalling that the synthetic evaluation σ_{igj} is determined by the modified fuzzy integral, $\sigma_{igj} = \int_F \alpha \partial \phi + \Delta^+ + \Delta^-$, it should be noted that, since $\phi(s_{pj})$, $\phi(s_{qj})$ are fixed for

²⁵ Recall that the product or service's quality q_{ig} is not accessible to the potential Consumer in this stage.

each Consumer j , $\phi(s_{pj} + s_{qj}) = 1$ and $\alpha_{pj} = f(p_{ig}, s_{pj})$, all variables involved are known by the time the Consumer is estimating σ_{igj} , with the exception of α_{qj} .

From Table 6.10, it can be seen that, whichever situation occurs, the basic synthetic evaluation σ_{igj}^* from a consummated purchase can only be one of the four values $\phi(s_{pj})$, $\phi(s_{qj})$, α_{pj} or α_{qj} ²⁶. But α_{qj} is inaccessible to the Consumer at this point, and even the *space* of possible alternatives cannot be completely described.

To surpass this restriction, a resort to an *Expected Value Interval* (EVI) of the payoff z_{igj} using *belief functions* [SHAF76] is proposed. The EVI method to be employed here is based in a paper by Thomas Strat [STRA90], and it was found to be appropriate to the present case, where fuzzy reasoning is being employed. This is so because a frequency or probabilistic original approach to the problem would require a greater number of data points in order to assess the chance of occurrence of each element from the frame of discernment (outcome space), which is not feasible to the Consumer as demonstrated above.

According to the theory [SHAF67], given a subset S contained in the space of outcomes Ω ($S \subseteq \Omega$), the *belief* in S is bounded by two limiting points: a lower limit, called *support*, denoted by $\text{Spt}(S)$, and an upper limit, the *plausibility*, represented by $\text{Pl}(S) = 1 - \text{Spt}(\bar{S})$ ²⁷.

Following the above definition, the payoff that would be ascribed to the buyer from a consummated transaction will be between the *support* and *plausible* values, that is, between $\text{Spt}(z_{ig})$ and $\text{Pl}(z_{ig})$. The method presented by Strat considers only these two extreme points, that are interpreted as the least and most favorable possible outcomes,

²⁶ It should be observed that it is not possible that a basic synthetic evaluation can be one of the importance factors $\phi(s_{pj})$ or $\phi(s_{qj})$ when the one associated to the greatest attractivity (α_{qj} or α_{pj}) is either greater than it or smaller than the smallest attractivity. In those circumstances, there will be only one type of adjustment in the basic fuzzy integral, either an increment, Δ^+ , or decrement, Δ^- , but not both at the same time.

²⁷ \bar{S} is the *complement* of S . Also, in most texts, the term *Belief* is used instead of *Support* denoting the lower extreme of the Belief interval. See Chapter 4 of this work - *A Review of Fuzzy Set Theory and Expert Systems* - and op. cit [STRA90].

respectively. In the sequence, a probability λ is assigned to the occurrence of the alternative which is most desirable to the decision maker²⁸.

In the present model, the meaning of λ shall be adjusted to the circumstances. Here, it will be assumed that λ , rather than a probability, is a prediction of the *degree of cooperation* of the supplier i in a particular iteration y of a cycle g of the game, mirrored by the item's quality $q_{ig} = f(c_{ig})$ (the method for the assessment of λ is explained ahead). Furthermore, the payoffs may lie anywhere in the interval bounded by $Pl(z_{igj})$ and $Spt(z_{igj})$. The analogy with the probabilistic approach is straightforward, since a greater value of λ implies in a bigger benefit, or payoff, to the Consumer²⁹.

To figure the predicted payoff, the consumer must find a point within $[Pl(z_{igj}), Spt(z_{igj})]$, with the aid of λ . Instead of mapping directly in that interval, a function associating $\lambda \rightarrow E^*(z_{ig})$ has been designed. For that operation, both pessimistic ($Spt(z_{igj})$) and optimistic ($Pl(z_{igj})$) assessments act like *attractors* for the desired point E^* , which is mirrored by the role that the ratio $\frac{(\lambda - Spt(z_{igj}))}{Pl(z_{igj}) - Spt(z_{igj})}$ plays in spotting $E^*(z_{igj})$ inside the interval delimited by $Spt(z_{igj})$ and $Pl(z_{igj})$.

The ratio $\frac{(\lambda - Spt(z_{igj}))}{Pl(z_{igj}) - Spt(z_{igj})}$ is taken relatively to 0.5, which stands for λ located at the middle point of the interval $[Pl(z_{igj}), Spt(z_{igj})]$. The resulting expression, that yields values in the range $[-1, 1]$, centered in zero for $\lambda = \frac{Pl(z_{igj}) - Spt(z_{igj})}{2}$, is a measure of the predicted *fulfillment of expectations* for the current iteration.

In this manner, the estimated payoff for the current iteration, denoted by $E^*(z_{ig})$, is given by Equations 6.20 (a, b).

²⁸ Although here it is not the case, because λ is settled by a rational agent, that is, the Firm, λ could also be interpreted as the *Nature's move*, in a situation of random occurrences not governed by preferences.

²⁹ Note that, even in the One-sided Prisoner's Dilemma, as the dichotomic choice C or D by the seller has been relaxed, its total cooperation, while extremely favorable to the buyer, implies in a zero unitary profit to the Firm. On the other hand, a continued and significant grade of *defection* implemented by the Firm will keep its potential customers refraining from consummating their purchases.

$$\text{IF } \left[\frac{(\lambda - \text{Spt}(\text{zig}))}{\text{Pl}(\text{zig}) - \text{Spt}(\text{zig})} \right] \geq 0.5$$

$$E^*(z_{ig}) = \left[\frac{2 \times (\lambda - \text{Spt}(\text{zig}))}{\text{Pl}(\text{zig}) - \text{Spt}(\text{zig})} - 1 \right] \times (\text{Pl}(\text{zig}) - \lambda) + \lambda, \quad [\text{Pl}(\text{zig}) - \text{Spt}(\text{zig})] > 0 \quad \text{Eq. 6.20(a)}$$

$$\text{IF } \left[\frac{(\lambda - \text{Spt}(\text{zig}))}{\text{Pl}(\text{zig}) - \text{Spt}(\text{zig})} \right] < 0.5$$

$$E^*(z_{ig}) = \left[\frac{2 \times (\lambda - \text{Spt}(\text{zig}))}{\text{Pl}(\text{zig}) - \text{Spt}(\text{zig})} - 1 \right] \times (\lambda - \text{Spt}(\text{zig})) + \lambda, \quad [\text{Pl}(\text{zig}) - \text{Spt}(\text{zig})] > 0 \quad \text{Eq. 6.20(b)}$$

When a new cycle begins, and the Consumer has not yet iterated with a particular Firm though, the fulfillment of expectations shall assume the value 1, meaning that the Consumer starts the cycle being thoroughly cooperative. This does not mean that it will buy the item, only that it is assigning to $E^*(z_{ig})$ the maximum value, that is, $\text{Pl}(z_{ig})$.

In the hypothesis that $\text{Pl}(\text{zig}) = \text{Spt}(\text{zig})$, $E^*(z_{ig})$ will be arbitrated as 1 or -1, depending on whether $\text{Pl}(\text{zig}) > 0$ or $\text{Pl}(\text{zig}) < 0$, respectively.

The outlook of the function $E^*(z_{ig}) = f(\text{Spt}(\text{zig}), \text{Pl}(\text{zig}), \lambda)$ is illustrated by the sample of curves of Figure 6.15.

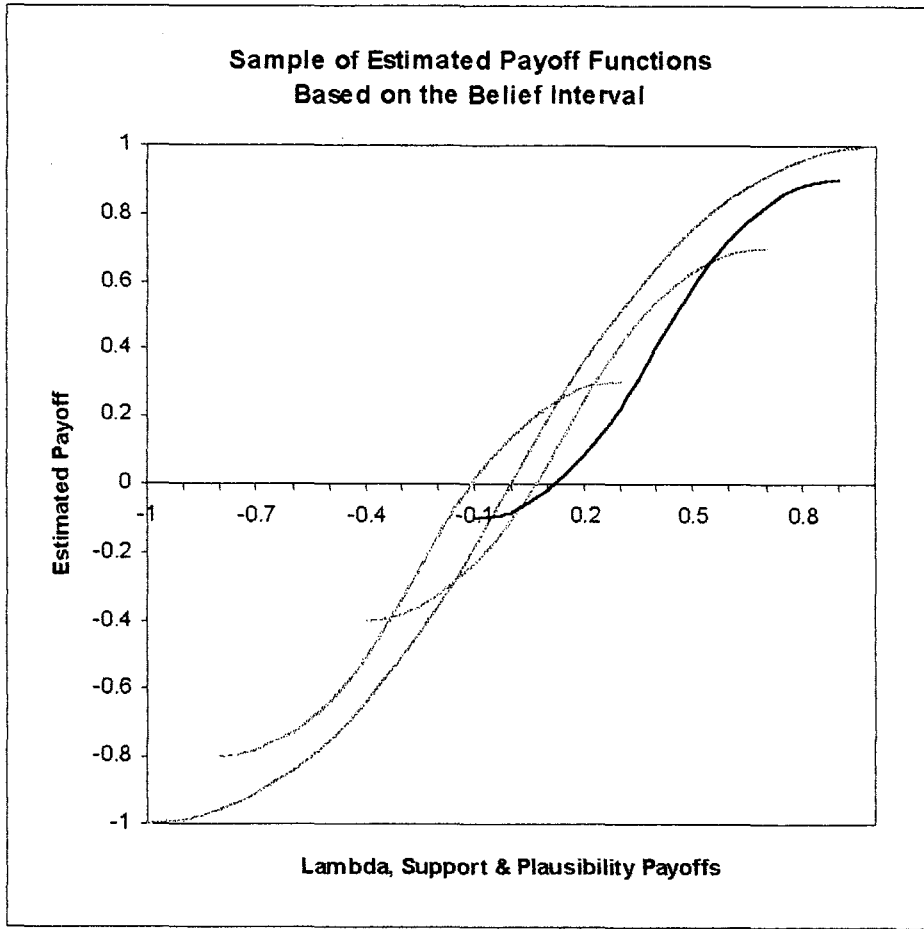


Figure 6.15 -Some Estimated Payoff Functions

Now, λ_{igj}^y is a measure of Firm i 's degree of cooperation as predicted by the Consumer j , in the iteration of order y of cycle g .

Analyzing Equations 6.16, 6.17, 6.18 and Table 6.10, it can be inferred that $Spt(z_{igj})$ should be determined employing the *most pessimistic* synthetic evaluation taken for the utmost worst case, that is, when $q_{ig} = 0$. Under this assumption, $\int_F \alpha^\nabla \partial \phi$ stands for the smallest possible value of σ_{igj}^0 when α_{pj} is known, where α^∇ corresponds to the attractivities for $\{p_{ig}, q_{ig} = 0\}$

Then, most pessimistic synthetic evaluation σ^{∇}_{igj} is given by:

$$\sigma_{igj} = \int_f \alpha^{\nabla} \partial \phi + \Delta^+ + \Delta^- \quad \text{Eq. 6.21}$$

Conversely, to find $Pl(z_{igj})$, α_{qj} is evaluated using the *best possible* prospect for the item's hypothetical quality q_{ig}^{30} . $Pl(z_{igj})$ will be calculated using σ^*_{igj} , which is the *most optimistic* synthetic evaluation. Equations 6.22(a) and (b) are used to compute σ^*_{igj} , where α^*_{qj} is the hypothetical *most optimistic* attractiveness of quality, which on its turn is calculated making $c_{ig} = p_{ig}$ in Equation 6.6.

$$\text{If } \alpha^*_{qj} \geq \alpha_{pj} \Rightarrow \sigma^*_{igj} = \max\{\alpha_{pj}, \min(\alpha^*_{qj}, \phi(s_{qj}))\} + \Delta^+ + \Delta^- \quad \text{Eq. 6.22(a)}$$

$$\text{If } \alpha^*_{qj} < \alpha_{pj} \Rightarrow \sigma^*_{igj} = \max\{\alpha^*_{qj}, \min(\alpha_{pj}, \phi(s_{pj}))\} + \Delta^+ + \Delta^- \quad \text{Eq. 6.22(b)}$$

Hence, both $Spt(z_{igj})$ and $Pl(z_{igj})$ can be determined using the appropriate equation from Table 6.13 replacing the argument σ_{igj} by σ^{∇}_{igj} and σ^*_{igj} , respectively, and also employing the specific Consumer's sensitivity s_{pj} .

However, the parameter λ^{∇}_{igj} is still unknown, and therefore it will be predicted with basis on the available previous data, both from the Consumer's own experiences and from the population likewise. Here, the population will be represented as a *typical* Consumer in two distinct ways:

- a) By the *median* of the sensitivities parameters of all the Consumers, which corresponds to $s_{pj}=0$, $s_{qj}=0$, as established by the model. This form is used to determine the *importance* to be allocated to the second type of representation.

³⁰ The "most optimistic" presupposition can also be dubbed unrealistic under a Firm's practical point-of-view, as long as it corresponds to $c_{ig}=p_{ig}$. But here this conjecture is admitted, since the *plausible*, not the *workable*, payoff is being investigated.

- b) By the *average* payoff obtained during the iterations of all members of the population with a specific Firm i in a cycle of the game.

The forecast of λ_{igj}^y shall be accomplished using for each source of information (Consumer and population), a relationship between earlier payoffs.

The weight or importance that a Consumer assigns to its own experience—in terms of payoffs received—and that derived from the population are assumed as different: The relevance of the latter will be taken as proportional to the *similarity* between itself and the typical Consumer described in the representation (a) above.

The similarity will be measured in a scale 0 to 1 and postulated as the proximity of sensitivities, expressed by $1 - \frac{|s_{qi} \text{ or } s_{pj}|}{5}$. In this manner, the nearer s_{pj} (or s_{qi}) is from 0, the more a particular Consumer and the typical one are considered alike, and the significance assigned to the population's data—the average payoff z_i —will tend to 1. On the other hand, the weight regarding a Consumer's own experience shall be *always* taken equal to 1. Then, the Firm's *degree of cooperation* λ_{igj}^y is plugged in Equation 6.20.

The resulting expression for λ_{igj}^y is depicted in Equation 6.23³¹.

$$\lambda_{igj}^y = \frac{\text{avg } z_{ig} \times \left(1 - \frac{|s_{qj}|}{5}\right) + z_{igj}^{y-1}}{2 - \frac{|s_{qj}|}{5}}, \quad \text{Eq. 6.23}$$

where:

- z_{igj}^{y-1} : payoff obtained by the Consumer j with Firm i , in the correspondent iteration of order $y-1$ of cycle g ³²;

³¹ Note that only the payoffs received by the Consumer as a result from accomplished purchases are considered, thus excluding the *zeros* that derived from eventual non-consummated purchases.

³² The current iteration being processed is the y^{th} .

- $\text{avg } z_{ig}$: average population's payoff in the transactions involving Firm i until the current moment, thus considering all accumulated results of the cycle, in terms of the Consumers' payoffs. It is calculated by Equation 6.25.

$$\text{avg } z_{ig} = \frac{\sum_j \sum_y z_{igj}^y}{\sum_j b_{igj}} \quad \text{Eq. 6.24}$$

The default values are $z_{igj}^{y-1} = 1.0$ and $\text{avg } z_i = 0.0$.

Now the payoff estimate $E^*(z_{ig})$ from Equation 6.20 can be thoroughly determined, and the threshold for the Consumer's decision will be established in the sequence.

II. The decision's threshold

The choice of a threshold for the Consumer's decision of buying or not the product or service in perspective was arbitrarily formulated in this model. Nonetheless, a fixed and constant value for this parameter, e. g. *zero*³³, has been avoided, because the idea is to allow a dynamic behavior for the buyers and consequently for the sellers. In this way, every Consumer will have a distinct and ever changing threshold value, influenced by how the game develops. Its determination will take into account each buyer's own knowledge base, along with some information gathered from the population's performance. Using this conception, the threshold τ_{gj}^y will be given by:

Case 1: IF a Consumer has not iterated yet (default value)

$$\tau_{gj}^y = 0.0 \quad \text{Eq. 6.25(a)}$$

³³ Implying that the Consumer would accept all deals with a predicted positive value.

Case 2: IF $\max\{\text{avg } z_g, (\max(z_{gj}^{y-3}, z_{gj}^{y-2}, z_{gj}^{y-1}))\} \leq 0$

$$\tau_{gj}^y = 0.0 \quad \text{Eq. 6.25(b)}$$

Case 3: IF $\min\{\text{avg } z_g, (\max(z_{gj}^{y-3}, z_{gj}^{y-2}, z_{gj}^{y-1}))\} > 0$

$$\tau_{gj}^y = \min\{\text{avg } z_g, \max(z_{gj}^{y-3}, z_{gj}^{y-2}, z_{gj}^{y-1})\} + \Delta^\tau \quad \text{Eq. 6.25(c)}$$

Case 4: IF

$$\min\{\text{avg } z_g, (\max(z_{gj}^{y-3}, z_{gj}^{y-2}, z_{gj}^{y-1}))\} < 0.0$$

and

$$\max\{\text{avg } z_g, (\max(z_{gj}^{y-3}, z_{gj}^{y-2}, z_{gj}^{y-1}))\} \geq 0.0$$

$$\tau_{gj}^y = \max\{\text{avg } z_g, \max(z_{gj}^{y-3}, z_{gj}^{y-2}, z_{gj}^{y-1})\} - \Delta^\tau, \quad \text{Eq. 6.25(d)}$$

where $\text{avg } z_g$ is the average payoff in the transactions involving all the Firms and the Consumer's Class, regarding all cycles, and z_{gj}^{y-1} , z_{gj}^{y-2} , z_{gj}^{y-3} are the last three payoffs received by the Consumer j from any Firm i , $i = 1, 2, \dots, m$, in the current cycle, excluding eventual zeroes due to non-consummated purchases.

In Equation 6.25(d), the amount $\Delta^\tau = [(\max(z_{gj}^{y-3}, z_{gj}^{y-2}, z_{gj}^{y-1}) - \mu_\tau)^2 + (\text{avg } z_g - \mu_\tau)^2]^{\frac{1}{2}}$,

where $\mu_\tau = \frac{\max(z_{gj}^{y-3}, z_{gj}^{y-2}, z_{gj}^{y-1}) + \text{avg } z_g}{2}$ or $\frac{\max(z_{gj}^{y-3}, z_{gj}^{y-2}, z_{gj}^{y-1})}{2}$ imitates the sample

standard deviation of the pair of values $\max(z_{gj}^{y-3}, z_{gj}^{y-2}, z_{gj}^{y-1})$ and $\text{avg } z_g$ when both are greater than zero, or only one is greater or equal to zero. Δ^τ represents an adjustment in the

required reference threshold. This operation has the objective of providing some flexibility for the Consumer's decision.

$$\text{avg } z_{ig} = \frac{\sum_i \sum_{\text{Class} \supset j} \sum_y z_{igj}^y}{\sum_i \sum_j b_{igj}} \quad \text{Eq. 6.26}$$

In accordance to this mechanism, the point of decision will be constantly updated in the course of the game, with the buyer trying to keep up with its best achievements, but also monitoring the general trend from the rest of the population. The default value for any missing value, a situation that will certainly occur in the beginning of the game, is *zero*.

The Consumer's final decision in the iteration will be the outcome from the comparison:

$$\text{If } E^*(z_{ig}) \geq \tau_{gj}^y \rightarrow \text{Buys}$$

$$\text{If } E^*(z_{ig}) < \tau_{gj}^y \rightarrow \text{Does not Buy}$$

At this point, an ensample with the application of the proposed method is opportune, so an example is provided.

Example

Suppose the Consumer from the previous example (p. 6.53) has not yet decided about the purchase of the item offered to it, so the quality is unobserved. Assume that all other variables and parameters remain the same. Additionally, consider that the following information from previous simulation rounds is available:

- $\alpha_{pj} = 0.4506$ (from page 6.54)
- $\text{avg } z_{ig} = 0.345$ (assumed);
- $z_{igj}^{y-1} = 0.234$ (assumed);

- $\text{avg } z_g = 0.342$ (assumed);
- $z_{gj}^{y-1} = 0.213$ (assumed);
- $z_{gj}^{y-2} = 0.354$ (assumed);
- $z_{gj}^{y-3} = \text{NA}$ (use default *zero*);
- $\sigma_{igj}^v = \alpha_{qj}^o \vee [\alpha_{pj} \wedge \phi(s_{pj})] + \phi(s_{pj}) \times \{\max [0, (\alpha_{pj} - \max (\alpha_{pj}^*, \max((\alpha_{qj}, \phi(s_{pj})))))]\} + \phi(s_{pj})^{\frac{1}{2}} \times \{\min [0, (\alpha_{pj} - \min (\phi(s_{qj}), \alpha_{qj}))]\}$
 $= 0.192437 + 0.04495328 - 0.02608$
 $= 0.2113$ (the *most pessimistic synthetic evaluation* of the item in perspective);

$\text{Spt}(z_{igj})$ is calculated using the adequate Equation from Table 6.13, in this case the one that corresponds to the range of $s_{pj} = 1.302$, that is, [1,2), replacing σ_{igj} by σ_{igj}^o .

$$\text{Spt}(z_{igj}) = \frac{0.2113 - 0.14132 + 0.004739 \times 1.302}{0.224981 + 0.005227 \times 1.302} - 1 = -0.671$$

As stated in the definition, $\text{Spt}(z_{igj})$ is the lower bound of the interval where the Consumer *believes* its payoff will lie; in other words, $\text{Spt}(z_{igj})$ is the buyer's most pessimistic estimate of the payoff for the current iteration.

Now, the upper limit $\text{Pl}(z_{igj})$ needs to be determined.

Making $c_{ig} = p_{ig} = 6.78$ yields (Eq. 6.6) $q_{ig} = 8.54$;

From Equations 6.13(c) and (d), using $s_{qj} = -1.302$, the most optimistic quality $q_{ig} = 8.54$ is qualified as:

$$u_{MD}(q_{ig} = 8.54) = 0.058;$$

$$u_{HG}(q_{ig} = 8.54) = 0.87;$$

To obtain α_{qj}^* , rules 2 and 3 from Table 6.9 are fired, resulting in:

$$u_{NT}(\alpha_{qj}^*) = 0.058 \text{ and}$$

$$u_{HG}(\alpha_{qj}^*) = 0.87.$$

After the defuzzification procedure (Figure 6.11), comes:

$$\alpha^*_{qj} = 0.724 \text{ (the most optimistic attractiveness of quality).}$$

Since $\alpha^*_{qj} > \alpha_{pj}$, Equation 6.22(a) is applied, yielding:

$$\begin{aligned} \sigma^*_{igj} &= \max\{0.4506, \min(0.724, 0.239039)\} + 0.239039 \times (0.724 - 0.4506) \\ &= 0.51595 \end{aligned}$$

Again, resorting to Table 6.13, $Pl(z_{igj})$ is computed using σ^*_{igj} as the substitute for the argument σ_{igj} .

$$Pl(z_{igj}) = \frac{0.51595 - 0.14132 + 0.004739 \times 1.302}{0.224981 + 0.005227 \times 1.302} - 1 = 0.64289$$

The prediction of the supplier's degree of cooperation λ^y_{igj} is defined by Equation 6.23.

$$\lambda^y_{igj} = \frac{\text{avg } z_{ig} \times \left(1 - \frac{|s_{qj}|}{5}\right) + z_{igj}^{y-1}}{2 - \frac{|s_{qj}|}{5}} = \frac{0.345 \times 0.7396 + 0.234}{1.7396}$$

$$\lambda^y_{igj} = 0.2812$$

Resourcing to Equation 6.20 (a), comes:

$$E^*(z_{ig}) = \left[\frac{2 \times (\lambda^y_{igj} - \text{Spt}(z_{igj}))}{Pl(z_{igj}) - \text{Spt}(z_{igj})} - 1 \right] \times [(Pl(z_{igj}) - \lambda^y_{igj})] + \lambda^y_{igj}$$

$$E^*(z_{ig}) = \left[\frac{2 \times (0.2812 + 0.671)}{0.643 + 0.671} - 1 \right] \times [0.643 - 0.2812] + 0.2812$$

$$E^*(z_{ig}) = 0.4438$$

The value $E^*(z_{ig}) = 0.4438$ matches the Consumer's expectations about the payoff it might receive if it buys the item. Now, this decision will be made comparing 0.1706 with the current threshold τ_{gj}^y , determined employing Equation 6.25(c).

$$\tau_{gj}^y = 0.342 + 0.00849$$

$$\tau_{gj}^y = 0.3505$$

Since $E^*(z_{ig}) \geq \tau_{gj}^y$ ($0.4438 \geq 0.3505$) \Rightarrow **The Consumer buys the item**

The presumed reason for that conclusion is that the Consumer gives credence to the possibility of accomplishing a good deal, based on its particular preferences and on the evidence it gathered from the foregoing process. Though, it should be remembered that the game has a highly dynamic characteristic: Recall that the payoff that are assigned to other Consumers in the event of succeeded purchases will obviously affect the variables that compound both λ_{igj}^y and τ_{gj}^y , crucial in its decision process. It may well be the case that its punctual verdict about buying (or not) from this very same Firm gets reversed in a future iteration.

6.5 - Summary of the Market Share Game

In this section, a synopsis of the developed model will be presented. The main concepts and variables used are highlighted in order to allow a clearer comprehensive insight of the process. In this manner, a review of the salient aspects contained in the model will be supplied.

a) Pre-play settings:

- Number of Firms (m);
- Fixed part of the probability of any Firm being selected in an iteration (s);

- Fixed parts of the Firms' negative payoffs—Penalty parameters t_1 and t_2 ;
- Number I of iterations per cycle (min $I = 3000$)

b) Initialization of Firms:

For each Firm i , $i = 1, 2, \dots, m$, the ensuing variables shall be defined for every cycle:

- Price p_{ig} ;
- Cost c_{ig} ;
- Advertising budget a_{ig} ;
- Penalty parameters t_1 and t_2 ;

The values assigned to those variables are valid only during a cycle of the game. After the completion of the first cycle, the simulation program will halt, and a opportunity to continue the game is offered to the user. If another cycle is to be run, p_{ig} , c_{ig} and a_{ig} may be modified.

c) The Consumers' Data

A set of information for each Consumer j is required, containing the elements listed in the sequence.

Identification of the Player and fixed Parameters:

- Serial Number j , $j = 1..1000$ (Section 6.4.5- I);
- Sensitivity Parameters (Table 6.6)

$$s_{pj} =$$

$$s_{qj} =$$

- Importance Factors (Eq. 6.16(a) and (b));

$$\phi(s_{pj}) =$$

$$\phi(s_{qj}) =$$

Iterations' Data:

- Payoff achieved in the last iteration with each of the m firms (Section 6.4.8- I);

$$Z_{ig}^{y-1} =$$

- Last three payoffs obtained in a cycle g of the game, with any Firm i (Section 6.4.8- II);

$$Z_g^{y-3} =$$

$$Z_g^{y-2} =$$

$$Z_g^{y-1} =$$

- Current decision threshold (Section 6.4.8- II);

$$\tau_{gj}^y =$$

Besides the information base mentioned above, which concerns every particular Consumer j , additional figures with respect to the population as a whole are necessary for the buyers' decision process. Likewise the *Iterations' Data*, the following variables are constantly being updated along the simulation process, and shared by all players.

- Average population's payoff in all the transactions involving each of the Firms i separately and accumulated through a cycle of the game (Eq.6.24);

$$\text{avg } Z_{ig} =$$

- Average population's payoff in all the transactions involving *all* m Firms, accumulated through *all cycles* of the game (Eq.6.26);

$$\text{avg } Z_g =$$

d) The Simulation Process

Before the simulation starts, the pre-play settings are defined and the Firms are initialized. The following steps describe the most important procedures and events that occur in the simulation process.

Step 1: Determine for each Consumer, its fixed parameters s_{pj} , s_{qj} , $\phi(s_{pj})$, $\phi(s_{qj})$;

- Step 2:** Select a Consumer j at random from its population;
- Step 3:** Select a Firm i according to the probability distribution defined by Equation 6.1;
- Step 4:** Characterize the price p_{ig} for the item being sold by the Firm i chosen according to the particular Customer's perception;
- Step 5:** Determine the Consumer's attractiveness of price α_{pj} (Table 6.8);
- Step 6:** Compute the Consumer's *most optimistic* attractiveness of quality α_{qj}^* (Equation 6.6, with $c_{ig} = p_{ig}$ and Table 6.9);
- Step 7:** Calculate the Consumer's *most pessimistic* and *most optimistic* synthetic evaluations σ_{igj}^V and σ^*_{igj} , respectively (Equations 6.21 and 6.22);
- Step 8:** Determine the support and plausible values $Spt(z_{igj})$ and $Pl(z_{igj})$ for the payoff (Table 6.13);
- Step 9:** Estimate the Firm's *degree of cooperation* λ_{igj}^y (Equation 6.23);
- Step 10:** Compute the Consumer's estimated payoff $E^*(z_{ig})$ (Equation 6.20);
- Step 11:** Determine the Consumer's threshold τ_{gj}^y (Equation 6.25);
- Step 12:** The Consumer decides to buy or not the item by comparing $E^*(z_{ig})$ and τ_{gj}^y ;
- Step 13:** A payoff is assigned to the Consumer: either *zero* or the value z_{igj}^y defined by Equation 6.18 and Table 6.13;
- Step 14:** A payoff is assigned to the Firm (Table 6.14);
- Step 15:** The bookkeeping of the results obtained in the iteration is made;
- Step 16:** Check if the number of iterations I established for the cycle has been reached; if not, return to Step 2; if yes, the simulation process halts and

may either be continued, starting a new cycle of the game with new values for p_{ig} , c_{ig} , a_{ig} and t_1 , t_2 , or terminated.

e) Bookkeeping and analysis of the results

The records of the results obtained in the simulation process will be the source of the analysis of the market share game. The criteria adopted for this important feature of the model, as well as the investigations to be performed on the output data and the respective conclusions reached consist the body of Chapter 7 of this work.

Chapter 7

Simulations of the One-sided Fuzzy IPD Market Share Game

7.1 - The Simulation Program

A special program was written to perform a sample of simulations of the model developed as a practical application of the FIPD, detailed in the Chapter 6. The need for developing a specific code derived from the fact that, in spite of the existence of a significant diversity of software platforms commercially available, none had the committing flexibility to allow its employment in the current case, given the specific conditions required.

The programming language elected was C++¹, mainly because of its low-level capabilities, widespread use and object-oriented features. Also, the code has been prepared to run under DOS, because no improved interface (e.g. Windows) was considered necessary, only the achieved results. In this manner, after running a game, that could consist of one or more *cycles*, the output, composed of *snapshots* of the partial results achieved during runtime, is saved in a tab-delimited text file, ready to be imported and manipulated in a higher level platform. This has been done with a spreadsheet (Excel), where the data was analyzed.

While the population of Consumers remained constant (1000 elements), the number of Firms could vary from 1 to 6, each with its own particular values for their variables².

The basic steps performed by the simulation program are depicted by the diagram shown in Figure 7.1.

¹ Borland C++, version 4.52, 1995.

² Recalling the Firms variables: Cost, Price, Advertising Budget and penalty parameters t_0 and t_1 .

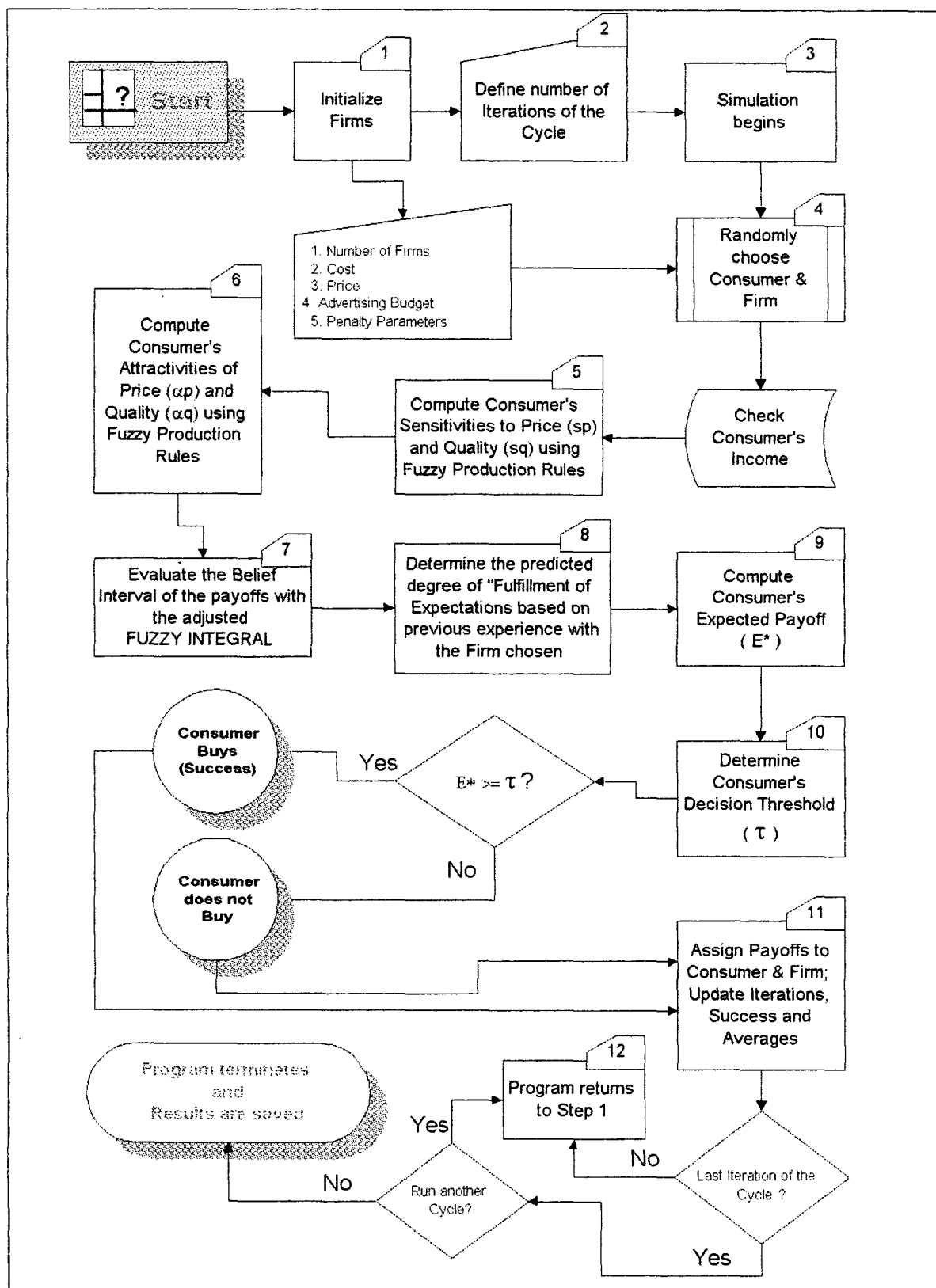


Figure 7.1 - Basic Diagram of the Simulation Process

The tuning of the model was accomplished by experimentally running preliminary simulations, which added up to about 1.200.000 iterations. The purpose of this Phase was observing the results and checking if they were reasonably compatible with a rational behavior, under the light of the personal differences in preferences on the part of the Consumers. The elements involved in the tuning stage were:

- The importance factors (ϕ_{spj} , ϕ_{sqj});
- Functions for the assignment of the Consumers' payoffs;
- Decision thresholds of the Consumers;
- The Firms' *degree of cooperation* (λ), and *fulfillment of expectations*.

One aspect of the game to be recalled is that the buyers do not compete among themselves, and, as the model incorporated a one-sided Prisoner's Dilemma, it is the duty of the Firms to adjust their behavior to the circumstances.

7.2 - Methodology of the Experiments

7.2.1 - Bookkeeping of the results

A mostly significant aspect that had to be taken care of, was the bookkeeping of the results generated by the simulation program. As previously mentioned, the bookkeeping was implemented by means of periodic snapshots of the current status of the players. The snapshots are taken by the program at different intervals along the iterations, depending on the Class to which the chosen Consumer belongs. They record the following data, which is done independently for every cycle of the game:

1. The Consumers' Classes (1 to 10, with the general Class 11 representing the whole population);
2. The Firms that took part in the iterations with each Class (0 to 5), with the general Firm 7 representing the collection of all Firms;
3. The number of encounters for each pair randomly selected (Class-Firm);
4. The number of successful iterations per pair;
5. The Consumers' accumulated payoff per pair;
6. The Firms' accumulated payoff per pair;
7. The serial number of the iteration referred to the cycle

7.2.2 - Scope of the Simulations

The possible combinations of ⟨Number of Firms⟩, ⟨ Cost ⟩, ⟨ Price ⟩, ⟨Advertising Budget⟩, ⟨ Penalty Parameters ⟩ and ⟨ Number of Cycles ⟩ to be simulated are vast, *in fact infinite*. Furthermore, due to the model structure, it makes a big difference if a specific set of values for the Firms' parameters is offered to the Consumers in the first cycle or in a subsequent one. In the latter case, the historic from previous cycles is likewise important. For those reasons, and also taking into account that the primary goal of the present research is the proposal of a new methodological approach to the modeling of conflict of interest using the Prisoner's Dilemma Paradigm, a limited number of experiments have been performed.

It is important to emphasize that the parameters employed in the model do not originate from any set of collected or empirical data, neither from other theoretical studies extracted from knowledge of Consumers' demeanor. Nevertheless, the parameters have been carefully selected (and tuned), always having in mind a plausible conduct of Consumers under the circumstances considered.

The cases studied with the simulation program consisted of four approaches (Phases). In Phases I, II and III, the probability of an encounter Consumer-Firm was made the same for every Company (equivalent advertising budgets). In Phase IV the Firms with the smallest ratio of success/iterations obtained in Phase III promoted a boost in their advertising budgets, and the tradeoff between benefits and additional advertising costs is analyzed. The four Phases are described next.

Phase I. A general competition including six Firms, composed of one game with five cycles with 18000 iterations each. In the average, every member of the population had three encounters with every Firm. The costs were set at three levels—one equal pair for each level—, remaining constant throughout the cycles. The prices were initially arbitrated at two levels for each cost. After the first cycle, they decreased for three of the Firms, while the other three Companies kept their prices without alterations. The criteria for varying prices

was to decrease them by 20% of the initial (Cycle 1) absolute markup in each cycle. The inputs are shown in Table 7.1. The penalty parameters for the Firms were $t_1=0.0$ and $t_2=0.1$.

| | Cycle 1 | | Cycle 2 | | Cycle 3 | | Cycle 4 | | Cycle 5 | |
|------|---------|-------|---------|-------|---------|-------|---------|-------|---------|-------|
| Firm | Cost | Price | Cost | Price | Cost | Price | Cost | Price | Cost | Price |
| 0 | 1.00 | 1.50 | 1.00 | 1.50 | 1.00 | 1.50 | 1.00 | 1.50 | 1.00 | 1.50 |
| 1 | 1.00 | 2.00 | 1.00 | 1.80 | 1.00 | 1.60 | 1.00 | 1.40 | 1.00 | 1.20 |
| 2 | 4.00 | 5.00 | 4.00 | 5.00 | 4.00 | 5.00 | 4.00 | 5.00 | 4.00 | 5.00 |
| 3 | 4.00 | 5.50 | 4.00 | 5.20 | 4.00 | 4.90 | 4.00 | 4.60 | 4.00 | 4.30 |
| 4 | 7.00 | 8.50 | 7.00 | 8.50 | 7.00 | 8.50 | 7.00 | 8.50 | 7.00 | 8.50 |
| 5 | 7.00 | 9.00 | 7.00 | 8.60 | 7.00 | 8.20 | 7.00 | 7.80 | 7.00 | 7.40 |

Table 7.1 - Costs and Prices for the General Competition (Phase I)

The results obtained in this part of the simulations were the object of a more complete analysis and discussed under several aspects, as will be presented further in this Chapter.

Phase II. A tournament with nine games, each with one cycle running 30000 iterations, so that a buyer would meet a Firm five times, in the average. For the first group of games (Group A — games 1-5), the prices were kept fixed, and the costs varied. The criteria for altering costs was starting off with a specific profit margin for each Firm, shown in Table 7.2, and decrease it by $\frac{1}{5}$ of the initial markup in every new cycle, with the costs getting nearer the prices. For the other group (Group B — games 1- 4), the

initial profit margins varied between 700% (Firm 0) and 42.86% (Firm 5), being reduced thereafter in each subsequent game.

The costs and prices employed in this approach of the experiments are presented in Tables 7.2 and 7.3.

| Firm | Price | Game 1 | | Game 2 | | Game 3 | | Game 4 | | Game 5 | |
|------|-------|--------|------------|--------|------------|--------|------------|--------|------------|--------|------------|
| | | Cost | Markup (%) | Cost | Markup (%) | Cost | Markup (%) | Cost | Markup (%) | Cost | Markup (%) |
| 0 | 1.00 | 0.50 | 100 | 0.56 | 80 | 0.63 | 60 | 0.71 | 40 | 0.83 | 20 |
| 1 | 2.55 | 1.50 | 70 | 1.63 | 56 | 1.80 | 42 | 1.99 | 28 | 2.24 | 14 |
| 2 | 3.75 | 2.50 | 50 | 2.68 | 40 | 2.88 | 30 | 3.13 | 20 | 3.41 | 10 |
| 3 | 5.40 | 4.00 | 35 | 4.22 | 28 | 4.46 | 21 | 4.74 | 14 | 5.05 | 7 |
| 4 | 7.50 | 6.00 | 25 | 6.25 | 20 | 6.52 | 15 | 6.82 | 10 | 7.14 | 5 |
| 5 | 9.00 | 7.50 | 20 | 7.76 | 16 | 8.04 | 12 | 8.33 | 8 | 8.65 | 4 |

Table 7.2 - Phase II, Group A: Constant Prices & Varying Costs

| Firm | Cost | Game 1 | | Game 2 | | Game 3 | | Game 4 | |
|------|------|--------|------------|--------|------------|--------|------------|--------|------------|
| | | Price | Markup (%) | Price | Markup (%) | Price | Markup (%) | Price | Markup (%) |
| 0 | 0.10 | 0.80 | 700 | 0.60 | 500 | 0.40 | 300 | 0.20 | 100 |
| 1 | 1.50 | 3.30 | 120 | 2.85 | 90 | 2.40 | 60 | 1.95 | 30 |
| 2 | 2.50 | 5.50 | 120 | 4.75 | 90 | 4.00 | 60 | 3.25 | 30 |
| 3 | 4.00 | 8.0 | 100 | 7.20 | 80 | 6.40 | 60 | 5.60 | 40 |
| 4 | 6.00 | 9.6 | 60 | 9.00 | 50 | 8.40 | 40 | 7.80 | 30 |
| 5 | 7.00 | 10.0 | 42.86 | 9.60 | 37.14 | 9.20 | 31.43 | 8.80 | 25.71 |

Table 7.3 - Costs, Prices and Markups for Simulations of Games of Phase II, Group B

The same costs assigned for Firms 1, 2, 3 and 4 in the initial game of Group A shall be used in the simulations of Group B. An exception is made with those regarding Firms 0 and 5. For these two cases, the costs have been reduced, with two separate objectives: For the former Firm, to allow the participation, in the market, of buyers belonging to Class 1 (what did not happen with the values employed in the games of Group A, as will be shortly seen); For the latter, to make possible a larger profit margin.

Phase III. This approach consists of a tournament with six unrelated games, each with six firms competing in a particular segment of the market. All the games are composed of a single cycle. The objective of this Phase was to investigate how the ratio Price /Cost (which can be likened to the inverse of the *degree of cooperation*) of the Firms, might affect the Firms' success ratio, as well as their net profits. The inputs for this Phase are depicted in Tables 7.4(a) and (b). The prices and costs in every game have been scattered within several ranges, in order to discover which are the Consumers' preferred choices.

| Firm | Game 1 | | | Game 2 | | | Game 3 | | |
|------|--------|-------|------------|--------|-------|------------|--------|-------|------------|
| | Cost | Price | Markup (%) | Cost | Price | Markup (%) | Cost | Price | Markup (%) |
| 0 | 0.25 | 0.40 | 60 | 2.00 | 2.80 | 40 | 3.50 | 4.20 | 20 |
| 1 | 0.25 | 0.55 | 120 | 2.00 | 3.20 | 60 | 3.50 | 4.90 | 40 |
| 2 | 0.25 | 0.70 | 180 | 2.00 | 3.60 | 80 | 3.50 | 5.60 | 60 |
| 3 | 0.25 | 0.85 | 240 | 2.00 | 4.00 | 100 | 3.50 | 6.30 | 80 |
| 4 | 0.25 | 1.00 | 300 | 2.00 | 4.40 | 120 | 3.50 | 7.00 | 100 |
| 5 | 0.25 | 1.15 | 360 | 2.00 | 4.80 | 140 | 3.50 | 7.70 | 120 |

Table 7.4(a) - Phase III: Games 1 - 3

| Firm | Game 4 | | | Game 5 | | | Game 6 | | |
|------|--------|-------|------------|--------|-------|------------|--------|-------|------------|
| | Cost | Price | Markup (%) | Cost | Price | Markup (%) | Cost | Price | Markup (%) |
| 0 | 4.50 | 5.40 | 20 | 5.50 | 6.60 | 20 | 6.50 | 7.15 | 10 |
| 1 | 4.50 | 6.075 | 35 | 5.50 | 7.15 | 30 | 6.50 | 7.80 | 20 |
| 2 | 4.50 | 6.75 | 50 | 5.50 | 7.70 | 40 | 6.50 | 8.45 | 30 |
| 3 | 4.50 | 7.425 | 65 | 5.50 | 8.25 | 50 | 6.50 | 9.10 | 40 |
| 4 | 4.50 | 8.10 | 80 | 5.50 | 8.80 | 60 | 6.50 | 9.75 | 50 |
| 5 | 4.50 | 8.775 | 95 | 5.50 | 9.35 | 70 | 6.50 | 10.0 | 53.8 |

Table 7.4(b) - Phase 3: Games 4 - 6

Phase IV. Here, the simulations have been specifically conducted to observing the influence of different advertising budgets in the Firms' profits and the relation between purchases and iterations—*success ratio*. The Phase was divided into two games, with four Firms each. In Game 1, all Firms had the same expected frequency of encounters with Consumers, because of equivalent advertising expenses. In Game 2, the Companies modified this situation by altering their publicity costs. Because the criteria of chosen the participants and their variables is based on the results yielded by Phase III, it was found preferable to present the data used in this stage directly in Section 7.3.5, after the mentioned outcomes have been displayed and commented.

7.3 - Analysis and Discussion of the Results

7.3.1 - General Remarks

As will be shortly seen from the simulations' outputs, the lower end of the population is not very sensible to improvements in quality, and the Firms aiming at that Classes would not do much better if they boosted the quality of their items while

maintaining the same prices. When the prices get higher, the sellers start to lose customers from the lower Classes but gain buyers from the more affluent levels.

One important rule of the game states that no Firm can sell for a price that is inferior to the respective item's cost, and the Consumers are aware of this norm. Thus, while smaller prices are always attractive (in different intensities) to every member of the population, they also carry a message about the maximum quality that might be embedded in the product or service on sale, which turns out to be the *plausible* payoff, derived from the most optimistic evaluation of the item (cost=price).

As a matter of fact, lowering prices is not necessarily always a good policy, depending on the Class to which the item is aimed. The attainment of the maximum gain varies with every Class of Consumers, and also of the current competition. The Consumers' preferences are basically subordinate to the sensitivity to price, quality, and also on the nonlinear relations that result from the employment of the adjusted fuzzy integral.

One relevant aspect to be recalled is that the price of the product is completely visible from the very start of the evaluation process, and the quality, before a transaction is consummated, can be only predicted.

Figure 7.2 illustrates the prices ($\div 10$) that correspond to the plausible payoffs, as well as the ranges of the synthetic evaluation σ_{igj} for the values of s_{pj} . The term "optimum price" should be understood as the price that yields the greatest possible payoff to a particular Consumer, in the occurrence of the *plausibility* hypothesis³.

³ Although the actual payoff increases with a greater quality and a smaller price, during the synthetic evaluation process, the true quality of an item is transparent (cannot be seen beforehand). Hence, if a price deviates significantly from the optimum in that stage, the upper bound of the evaluation interval will become consequently lower, thus limiting the range of the predicted payoff to be achieved in the transaction.

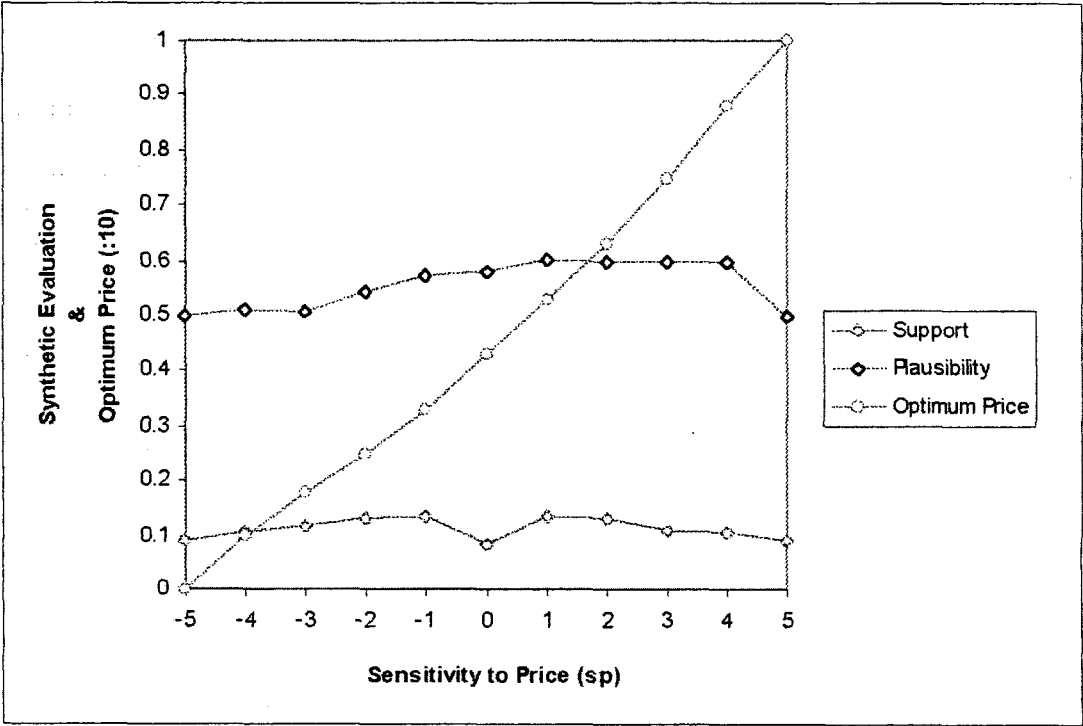


Figure 7.2 - Synthetic Evaluation Ranges and Optimum Prices as Functions of s_{pi}

7.3.2 - The Simulations of Phase I

In this Phase, no Consumers belonging to Class 1 (which has the lowest income) ever accomplished a purchase, with the minimum price set at MU\$1.50. This fact indicates an unattended and unexplored market niche concerning that category of buyers, that is rather numerous, but has great limitations regarding the purchasing power, together with a significant slack in the demand for quality. On the other hand, Classes 8, 9 and 10 had a rate of 100% of success in the iterations with Firms 4 and 5. Considering the relative insensitivity to price of the Consumers pertaining to this spectrum, that finding can be interpreted as a hint that the Firms selling to those members might obtain a higher profit if they increase their prices, what could be done associated to some expenses in advertising.

The graphs depicted in Figures 7.2(a)-(f) summarize the evolution of the Consumers' conduct along the iterations of Cycle 1. In this stage, the prices from Firms 1, 3 and 5 were the highest of the game. Because of the algorithm employed for the Consumers' decision, which always employed the maximum fulfillment of expectations in

the first iteration with any Firm in a given cycle, the rate of success for those Firms steadily decreased along the course of encounters. This situation is mainly due to the presence, of another Firm offering a more advantageous deal, in the same range of cost/price.

The average slope is different for each Class, as should be expected. For the Firms with the lower prices, a equilibrium seems to have been reached, in term of the percentage of successful iterations.

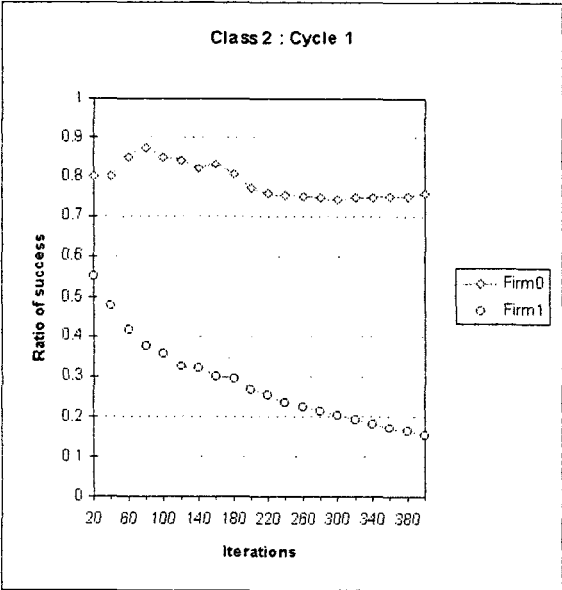


Figure 7.3(a)

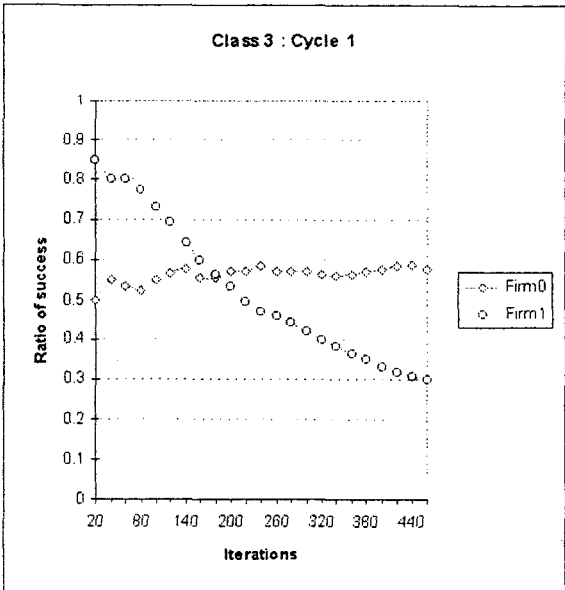


Figure 7.3(b)

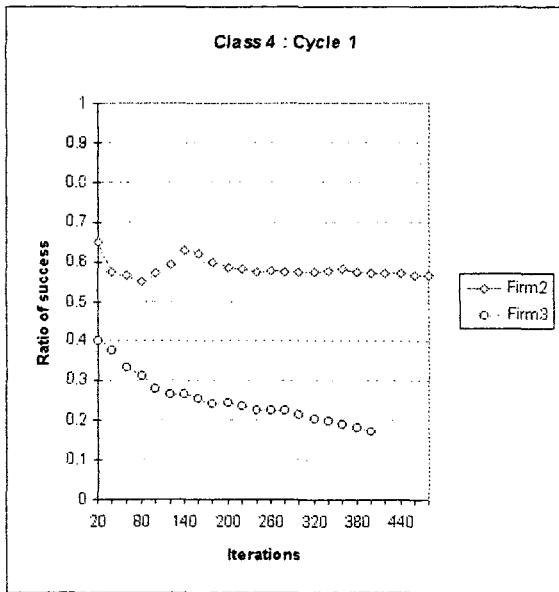


Figure 7.3(c)

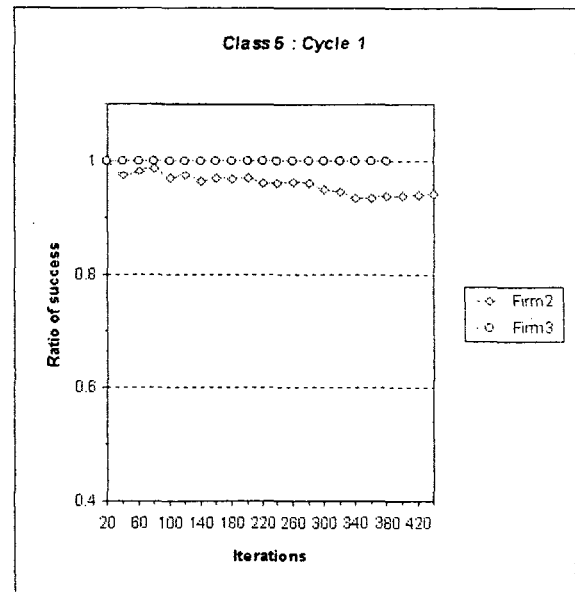


Figure 7.3(d)

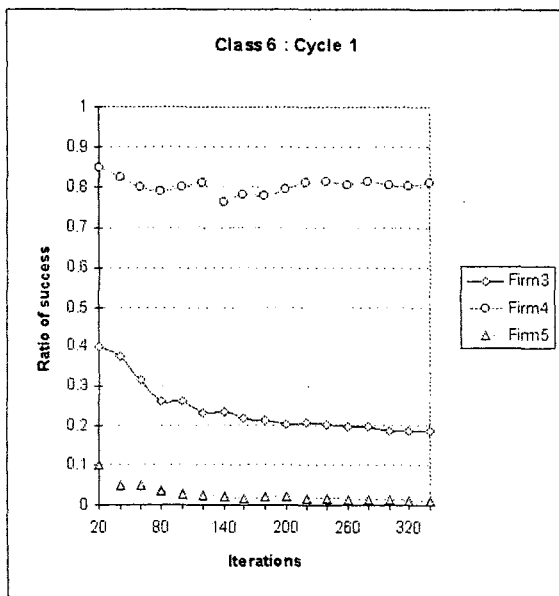


Figure 7.3(e)

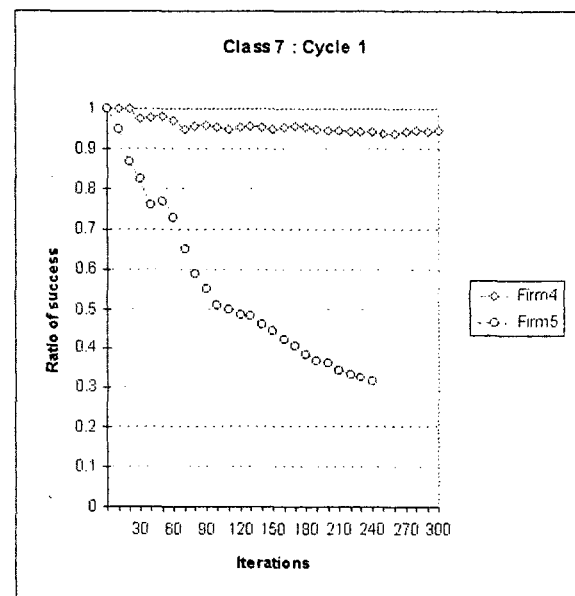


Figure 7.3(c)

The illustration in Figure 7.4 shows how the sales were distributed among the Classes in Cycle 1. Class 5, which is located around the median income, had the greatest participation in the market, in terms of number of purchases. This result is consistent with empirical observations of some real market profiles.

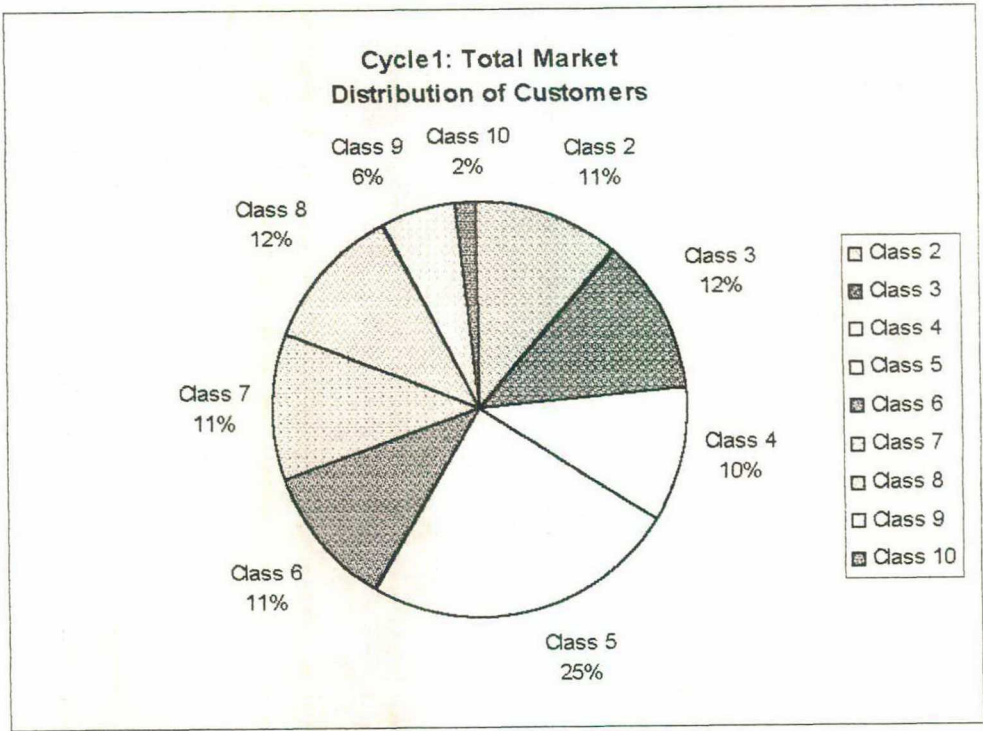


Figure 7.4 - Cycle 1: % of Total Successful Iterations with All Firms

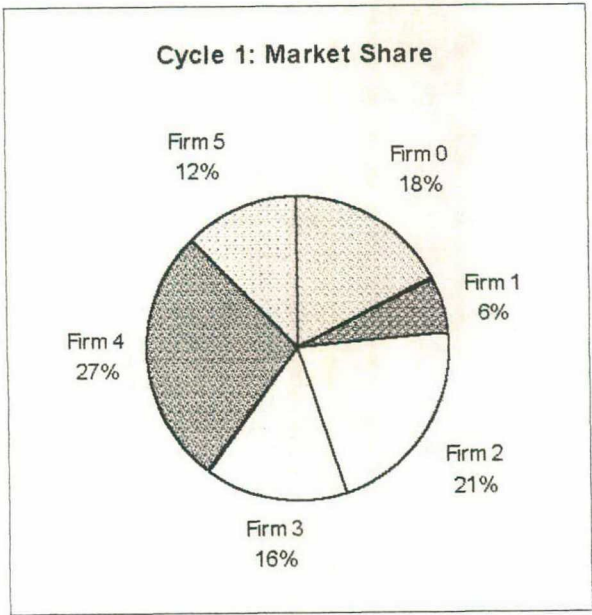


Figure 7.5 - Cycle 1: % of Successful Iterations achieved by the Firms with the Population

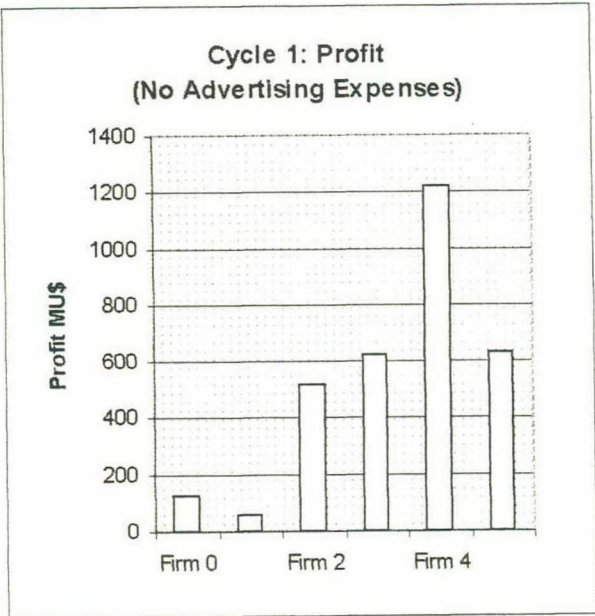


Figure 7.6 - Cycle 1: Absolute Values of Profit, per Firm

As can be seen from Figures 7.5, 7.6 and 7.7, Firm 4 had the best performance in Cycle 1, either in the rate of successful iterations or profit. The reason for this realization seems to lie on Firm 4's relative profit margin (21.43%), which was the smallest of all competitors. In subsequent cycles, Firm 4 lost the initial advantage it had during the iterations of Cycle 1, even though its cost and price had remained constant, which shows the effect of a fiercer competition, because its closest adversary, Firm 5, had lowered its price.

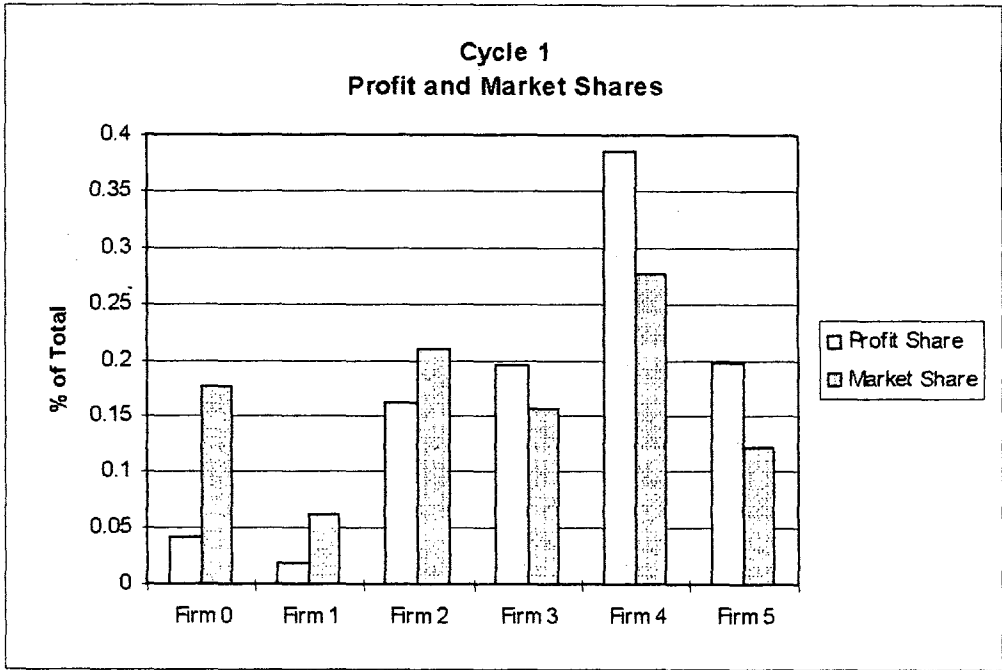


Figure 7.7 - Cycle 1: Comparison between Profit and Market Shares, as % of the Total

Figures 7.8 through 7.13 depict how the market evolved along the cycles of the game, separately for each Class of Consumers. The graphs show the information regarding the Firms from which the Consumers bought. As mentioned earlier, Class 1 did not buy from anyone, implying in the existence of an unattended niche. As to Classes 8, 9 and 10, which only bought from Firms 4 and 5, the rate of successful iterations was 100% for both Companies in every cycle except cycle 5, when the rate of success for Firm 5 fell to 0.622, 0.531 and 0.429, concerning Classes 8, 9 and 10, respectively. This result demonstrates the inadequacy of decreasing prices too much, mainly on the higher segment of the market. Figure 7.14 illustrates the information for the population as a whole.

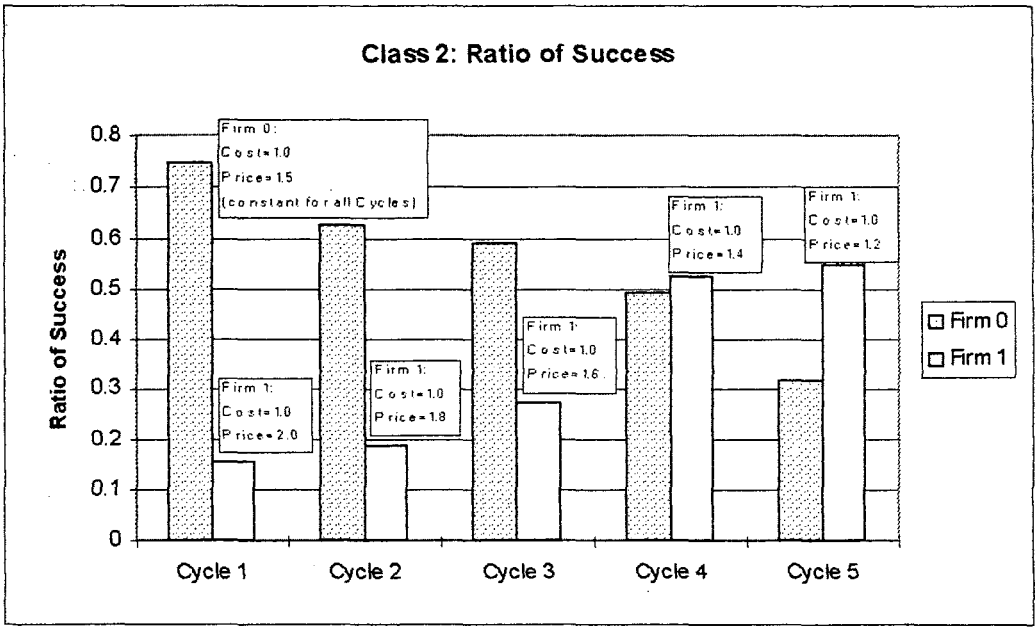


Figure 7.8 - Evolution of the Ratio of Success along the Cycles among Members of Class 2

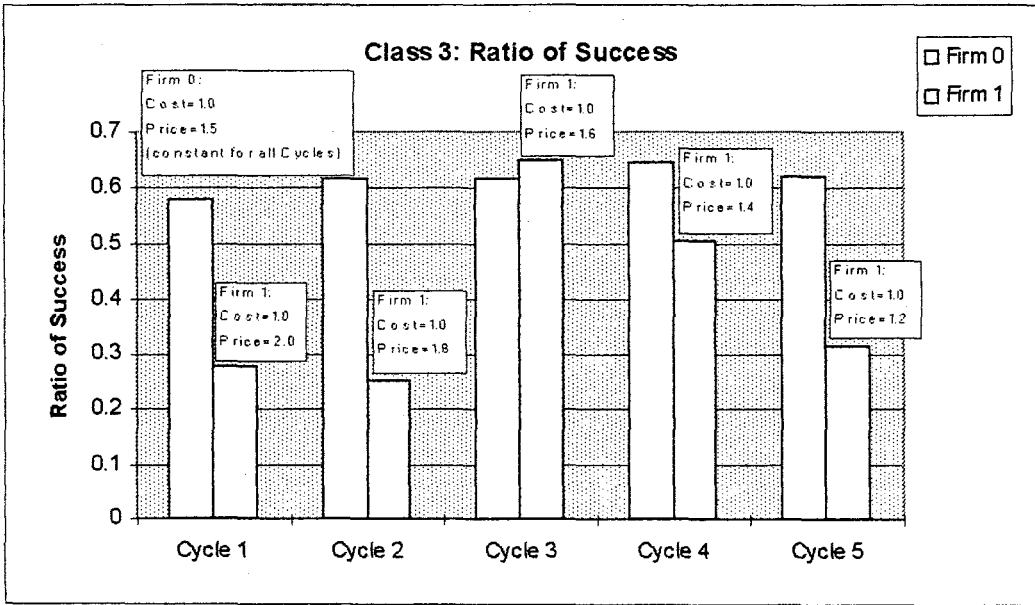


Figure 7.9 - Evolution of the Ratio of Success along the Cycles among Members of Class 3

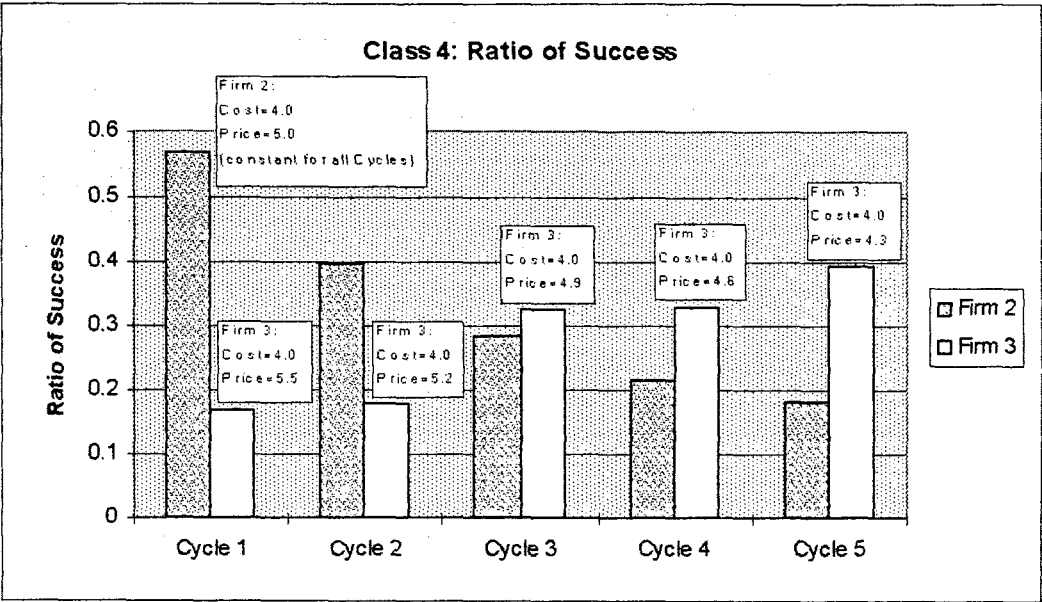


Figure 7.10 - Evolution of the Ratio of Success along the Cycles among Members of Class 4

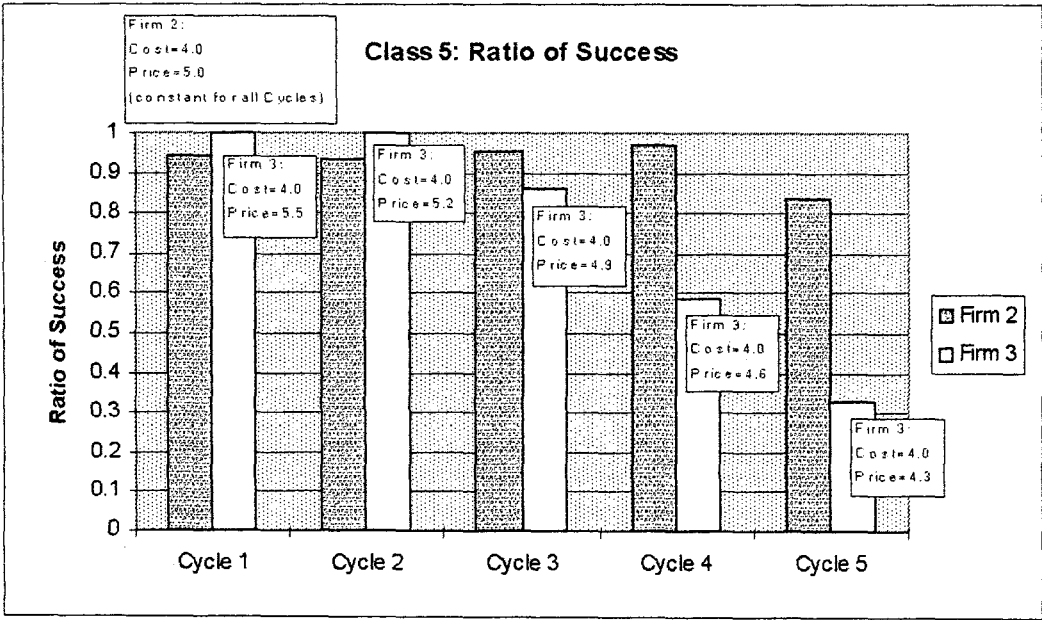


Figure 7.11 - Evolution of the Ratio of Success along the Cycles among Members of Class 5

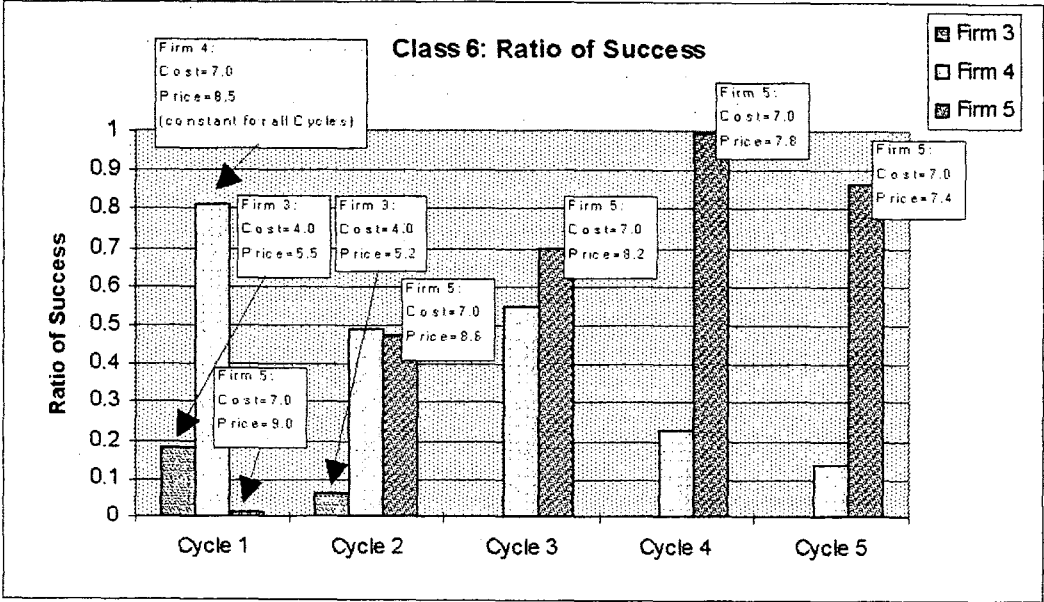


Figure 7.12 - Evolution of the Ratio of Success along the Cycles among Members of Class 6

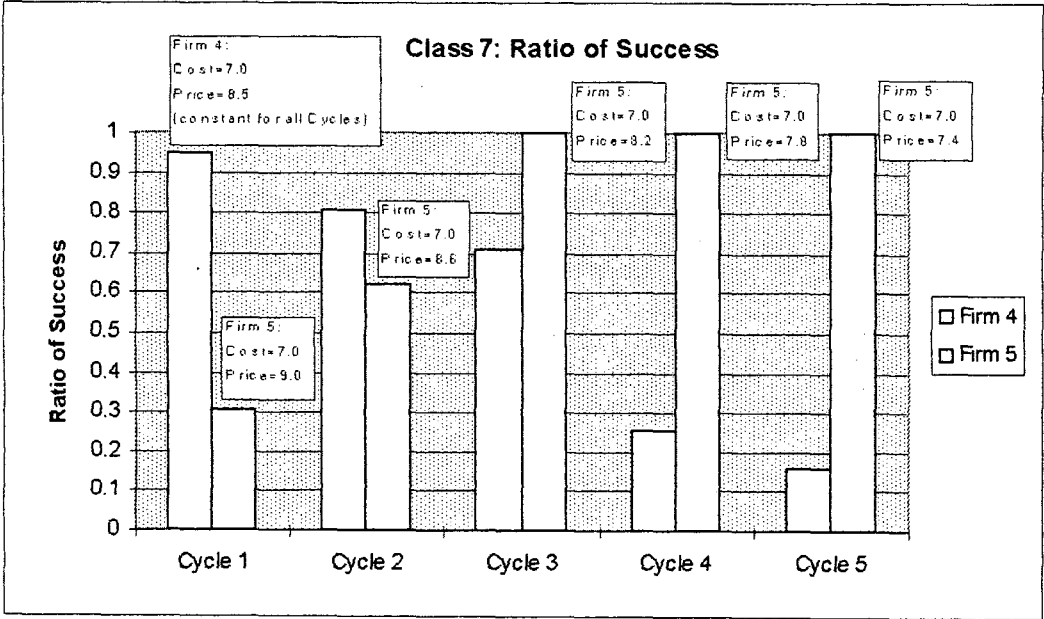


Figure 7.13 - Evolution of the Ratio of Success along the Cycles among Members of Class 7

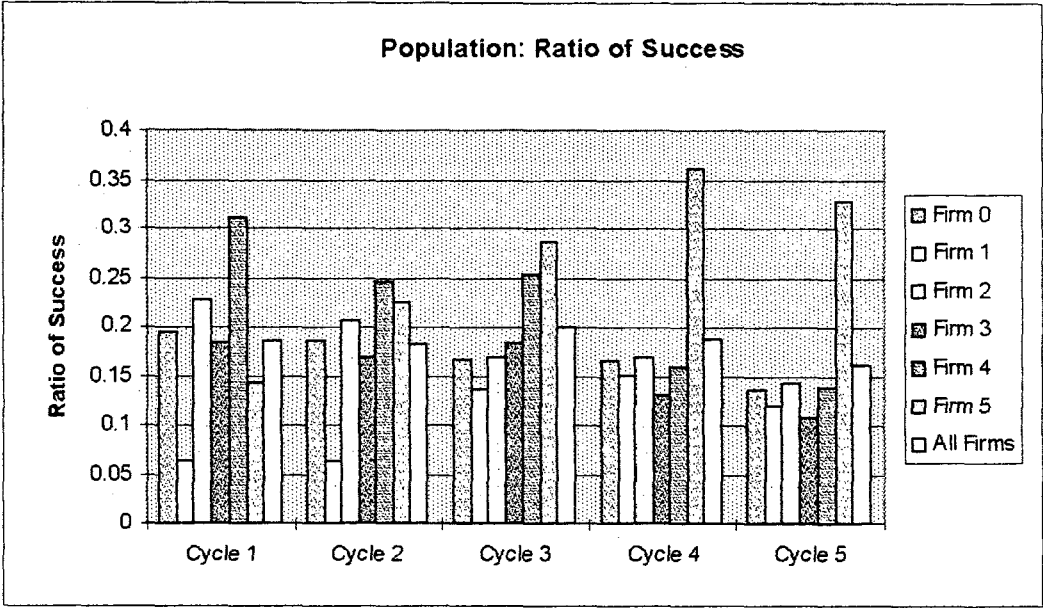


Figure 7.14 - Evolution of the Ratio of Success along the Cycles among Members of the Population

The graphs of Figures 7.15, 7.16 and 7.17 show the variation of the market share and profit of the pair of Firms that competed in the same range, as functions of the price. Note that only the Firms 1, 3 and 5 varied their prices, while the other even-numbered Companies maintained theirs charges constant.

In the lower end (Figure 7.15), the results clearly demonstrate that Firm 1 did not get any benefit by reducing its prices beyond about 1.6. Its local maximum profit occurs nearby this point. Even though there is some room for a market share gain (Firm 1), pushing prices further down only causes monetary losses to both competitors.

For the mid-range Companies, the same comments apply, with the difference that no local maximum occurred for Firm 3. A strategy that seems appropriate for the present and previous case is to experiment a slight increase in price, since the curves indicate that the profit could be larger in that direction.

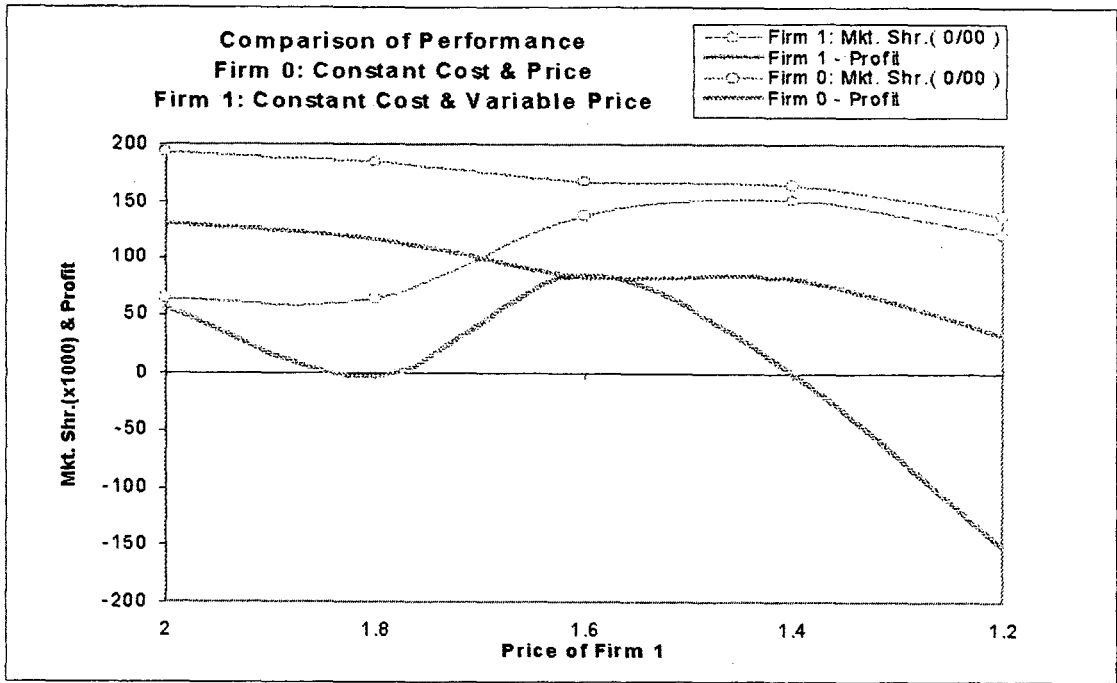


Figure 7.15 - Comparison of Global Profit and Market Share for Firms 0 and 1

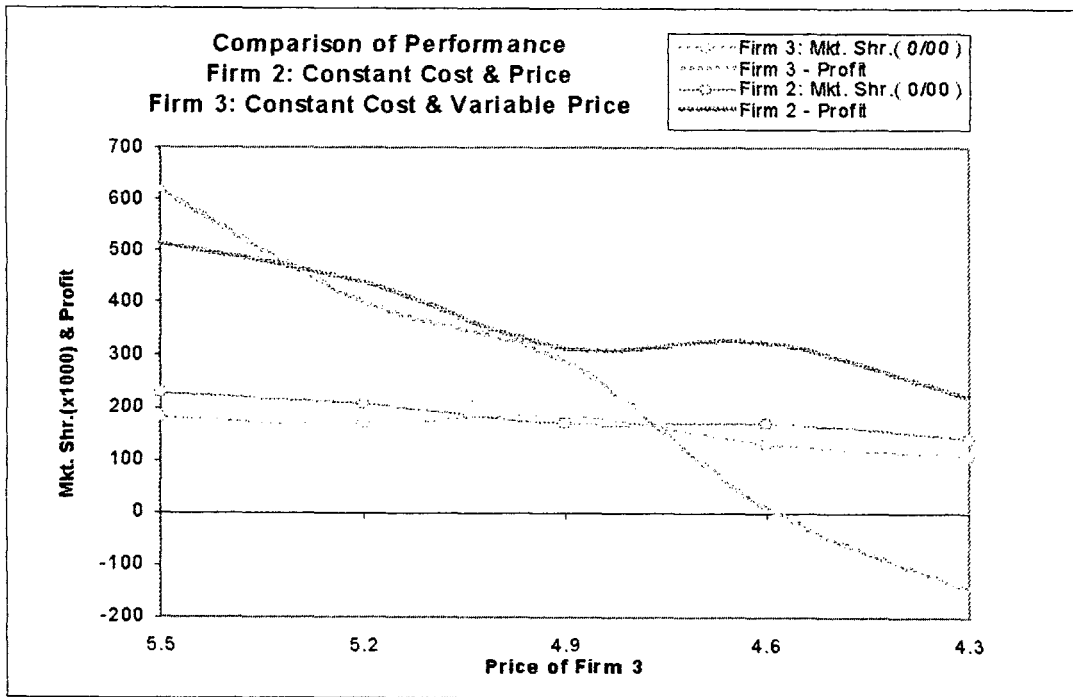


Figure 7.16 - Comparison of Global Profit and Market Share for Firms 2 and 3

In the higher end, the shape of the profit curve for Firm 5 attest that decreasing the price below around 8.6 is to no avail. Again, its market share raises, but the monetary result diminishes.

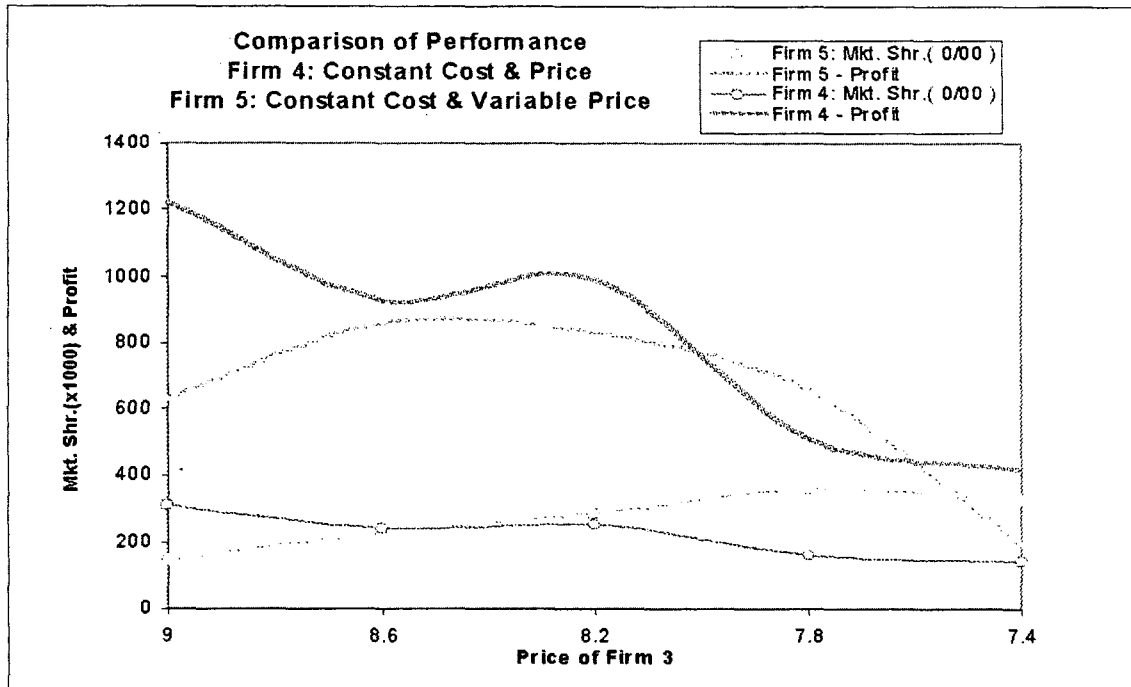


Figure 7.17 - Comparison of Global Profit and Market Share for Firms 4 and 5

Conclusions for Phase I of the Simulations

Recall that the even-numbered Firms remained static, serving as a kind of *benchmark* for the formulation of its closest competitor's policies.

- For the market aimed at the less affluent Consumers, competing by increasing quality and consequently cost, is not quite important— the specific buyers are somewhat insensible to that attribute. Moreover, here a small difference in price can be fundamental.
- As a general remark, reducing prices too much is not a good policy— this appears to bring about two different effects: For the initial Classes, some gain in the market

share occur, until a certain limit, but the profit normally plunges more abruptly. For the other extreme, since the potential buyers do not have knowledge about the real profit margin of the Firms, it looks like a suspicion about the quality embedded in the product arises, which causes a narrower interval for the item's anticipated evaluation.

- A greater emphasis in the quality should be a priority if a Firm plans to acquire a larger slice of the upper spectrum of Consumers. In this case, price should not be the object of competition.
- The results achieved by the Firms are completely interrelated, so it is wise not to “rock the boat”, in terms of starting a price war with the objective of gaining a larger number of buyers. In the case of sellers and buyers, if a Firm is too *cooperative* regarding the Consumers, it will appear *defective* in the eyes of the other Companies, which may cause retaliatory policies and collective losses.
- Although in this model there is no differentiation in quality due to technological attributes of the items being sold, this could be done by assigning diverse functions to the perception of quality based in the cost⁴.
- As mentioned earlier, the possibilities to combine costs, prices and advertising costs are endless. The purpose of the present group of experiments is to demonstrate the analytical capabilities of the model.

⁴ See Chapter 6, Section 6.4.4.

7.3.3 - The Simulations of Phase II

The analysis of the results of this Phase will be done independently for each Firm, because they do not compete (mainly) for the same niches of the market⁵. Furthermore, only those Classes which played a significant role on the Firms' results shall be examined.

a) Group A, Games 1 - 5: Variable (increasing) costs, constant prices

Firm 0: Price = 1.0

| Cost | Class 2: Succ. Ratio (%) | Class 2: Profit | Class 3: Succ. Ratio (%) | Class 3: Profit |
|------|--------------------------|-----------------|--------------------------|-----------------|
| 0.5 | 35.65 | 98.75 | 1.05 | -33.60 |
| 0.56 | 54.63 | 146.38 | 4.18 | -26.12 |
| 0.63 | 62.96 | 143.16 | 6.07 | -29.62 |
| 0.71 | 59.81 | 104.91 | 3.92 | -43.48 |
| 0.83 | 65.45 | 56.66 | 0.75 | -64.55 |

Table 7.5 - Results of Firm 0

For Firm 0, whose price was the lowest available to customers, the results indicate that an optimum point could be found within the interval covered by the simulations. The point where the cost was ≈ 0.58 yielded the best profit. The ratio of success continued to increase after that point, but it brought no additional monetary benefit for that Firm, since its absolute profit margin also got lower.

An interesting finding was that the Consumers from Class 1 could not find an appropriate deal from any seller, not even with the relatively low price 1.00 charged by Firm 1. Once again., it hints to the existence of an untended market niche in the lower spectrum of incomes.

Figure 7.18 shows the curves for the iterations performed between Firm 0 and Class 2 of Consumers. Class 3, which also initially accomplished a small percentage of purchases (Table 7.5), did not sustain its willingness to buy from Firm 0, preferring the

⁵ Nevertheless, the interdependency still prevails, thereby it may happen that some of the Consumers of a Class migrate to another, if the relation cost/price of the nearest competitor becomes more attractive.

deals offered by Companies 1 and 2, as illustrated next in Table 7.6, 7.7 and Figure 7.19.

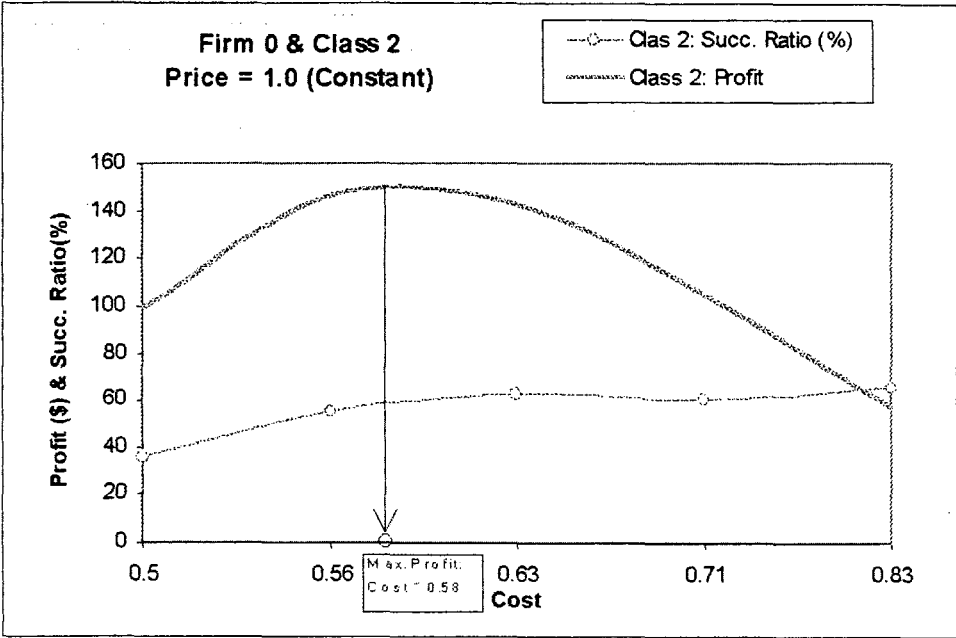


Figure 7.18 - Firm 0: Success Ratio and Profit as a Function of Cost

Firm 1: Price = 2.55

| Cost | Class 2: Succ. Ratio (%) | Class 2: Profit | Class 3: Succ. Ratio (%) | Class 3: Profit | Class 4: Succ. Ratio (%) | Class 4: Profit |
|------|--------------------------------|--------------------|--------------------------------|--------------------|--------------------------------|--------------------|
| 1.50 | 4.77 | -4.11 | 70 | 566.64 | 0.08 | 23.91 |
| 1.63 | 5.56 | -6.17 | 73 | 544.98 | 0.10 | 21.89 |
| 1.80 | 3.72 | -27.94 | 89 | 521.86 | 0.14 | 36.33 |
| 1.99 | 4.53 | -36.84 | 94 | 401.77 | 0.14 | 8.78 |
| 2.24 | 3.47 | -51.15 | 87 | 192.05 | 0.13 | -28.34 |

Table 7.6 - Results of Firm 1

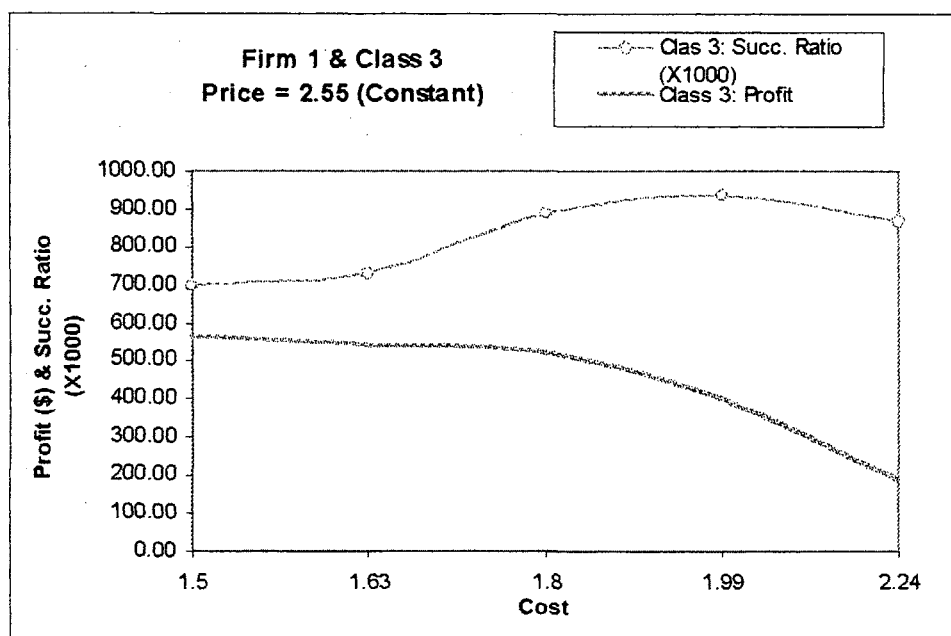


Figure 7.19 - Firm 1: Success Ratio and Profit as a Function of Cost

The Consumers that belong to Class 3 shared their preference almost equally between Firms 2 and 3. It can be seen that, in an apparent contradictory manner, that category of buyers decreased the rate of acquisitions from both Firms, when the cost embedded in the product raised.

To explain this presumable glitch, it must be recalled that the neither the evaluation function nor the decision threshold for the Consumers pertain exclusively to any particular pair Firm / Class. Instead the functions incorporate constantly updated figures, in runtime, that come from the iterations with the rest of the population with the same Firm— λ and the fulfillment of expectations—for the estimated payoff, and from other Firms with the same Class— τ (decision's threshold). Hence, better or worse deals that are taking place elsewhere are playing a role in the outcomes from the simulation program.

Table 7.7 shows the results achieved by Firm 2 with Classes 2 and 3. Considering that those two category of Consumers performed a significant number of successful

iterations with Company 2, the graph of Figure 7.20 illustrates the curves with the added outcomes.

Firm 2: Price = 3.75

| Cost | Class 2: Succ. Ratio (%) | Class 2: Profit | Class 3: Succ. Ratio (%) | Class 3: Profit |
|------|--------------------------|-----------------|--------------------------|-----------------|
| 2.50 | 20.25 | 158.98 | 76.96 | 763.85 |
| 2.68 | 15.39 | 83.30 | 82.60 | 650.90 |
| 2.88 | 21.47 | 96.60 | 84.39 | 578.52 |
| 3.13 | 9.68 | -12.06 | 88.82 | 416.28 |
| 3.41 | 6.52 | -51.09 | 75.56 | 176.73 |

Table 7.7 - Results of Firm 2

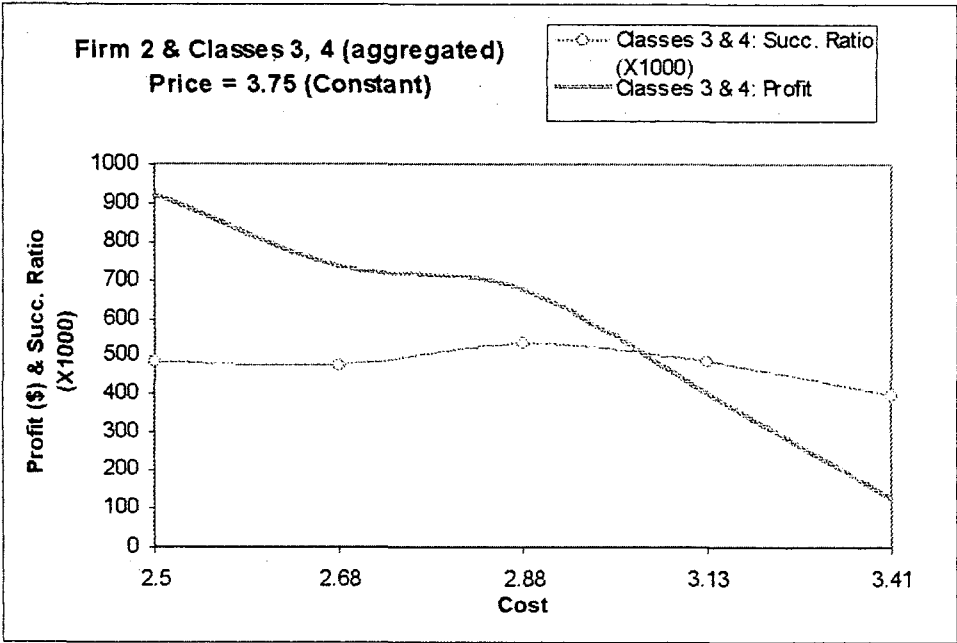


Figure 7.20 - Firm 2: Success Ratio and Profit as a Function of Cost

A point of maximum ratio of success can be identified around the cost = 2.88. But it is not associated to the maximum profit, given the payoff functions selected for the Firms. On the other hand, it appears that Firm 2 is not acting well— in the sense of optimizing

its profit— by improving the quality of the item. The shape of the profit curve hints that, on the contrary, a reduction of cost, with the price fixed, would be a better strategy to be adopted in this case.

The foregoing comment applies almost integrally for Firms 3 and 4, since the figures depicted in Table 7.8 and 7.9 are coherent with the stated presumption. However, a relevant difference between those two suppliers can be seen: While Firm 3 had its customers picked from Classes 4, 5 (the majority) and 6, Firm 4 had the bulk of its market niche distributed among the Consumers from Classes 6 and 7.

Firm 3: Price = 5.40

| Cost | Class 4 Succ. Ratio (%) | Class 4 Profit | Class 5 Succ. Ratio (%) | Class 5 Profit | Class 6 Succ. Ratio (%) | Class 6 Profit |
|------|-------------------------------|-------------------|-------------------------------|-------------------|----------------------------|-------------------|
| 4.00 | 33.81 | 327.55 | 100.00 | 905.81 | 16.3 | 98.91 |
| 4.22 | 36.12 | 279.20 | 100.00 | 807.12 | 18 | 87.33 |
| 4.46 | 27.90 | 159.11 | 100.00 | 637.32 | 14.5 | 39.05 |
| 4.74 | 25.57 | 76.86 | 100.00 | 440.88 | 19.49 | 33.90 |
| 5.05 | 13.88 | -22.35 | 95.48 | 226.35 | 18.47 | -6.67 |

Table 7.8 - Results of Firm 3

As can be clearly inferred from the tabulated results for Firm 4, the improvement in quality derived from the assignment of a higher cost to the item had very little impact in the market share concerning Classes 6 and 7, which appear to be already satisfied with the initial value. The same situation did not occur with Classes 8, 9 and 10, which show a trend of a larger ratio of success associated with increasing quality. Conversely, due to the reduction on the profit margin, Company 4 suffered untoward consequences in its profits. For the data employed in the runs of simulations, the best position was attained with the initial values (cost = 6.00, price = 7.50).

Firm 4: Price = 7.50

| Cost | Class 6: Succ. Ratio (%) | Class 6 Profit | Class 7 Succ. Ratio (%) | Class 7 Profit | Class 8 Succ. Ratio (%) | Class 8 Profit |
|-------------|---------------------------------|-----------------------|---------------------------------|------------------------|--------------------------------|-----------------------|
| 6.00 | 82.87 | 1262.59 | 96.71 | 1101.44 | 28.90 | 237.28 |
| 6.25 | 83.73 | 1184.93 | 96.92 | 1132.82 | 23.12 | 160.44 |
| 6.52 | 83.25 | 966.56 | 89.37 | 812.00 | 29.97 | 171.07 |
| 6.82 | 82.73 | 1013.11 | 80.08 | 840.63 | 32.66 | 213.53 |
| 7.14 | 80.51 | 680.54 | 75.63 | 483.85 | 32.71 | 146.03 |
| Cost | Class 9: Succ. Ratio (%) | Class 9 Profit | Class 10 Succ. Ratio (%) | Class 10 Profit | | |
| 6.00 | 30.22 | 128.54 | 19.61 | 22.11 | | |
| 6.25 | 27.51 | 106.93 | 14.49 | 18.16 | | |
| 6.52 | 28.64 | 101.97 | 28.21 | 19.63 | | |
| 6.82 | 21.34 | 61.23 | 27.66 | 23.80 | | |
| 7.14 | 34.38 | -65.86 | 30.61 | 17.54 | | |

Table 7.9 - Results of Firm 4**Firm 5: Price = 9.00**

| Cost | Class 6: Succ. Ratio (%) | Class 6 Profit | Class 7 Succ. Ratio (%) | Class 7 Profit | Class 8 Succ. Ratio (%) | Class 8 Profit |
|-------------|---------------------------------|-----------------------|---------------------------------|------------------------|--------------------------------|-----------------------|
| 7.50 | 2.74 | -23.42 | 65.77 | 461.75 | 100.00 | 451.50 |
| 7.76 | 4.79 | -14.19 | 69.63 | 405.21 | 100.00 | 395.56 |
| 8.04 | 3.28 | -33.42 | 50.35 | 186.59 | 100.00 | 312.00 |
| 8.33 | 4.09 | -38.95 | 39.15 | 96.81 | 100.00 | 203.01 |
| 8.65 | 0.84 | -54.96 | 24.15 | 5.49 | 97.95 | 99.52 |
| Cost | Class 9: Succ. Ratio (%) | Class 9 Profit | Class 10 Succ. Ratio (%) | Class 10 Profit | | |
| 7.50 | 100.00 | 244.50 | 100.00 | 87.00 | | |
| 7.76 | 100.00 | 210.80 | 100.00 | 60.76 | | |
| 8.04 | 100.00 | 161.28 | 100.00 | 54.72 | | |
| 8.33 | 100.00 | 102.51 | 100.00 | 33.50 | | |
| 8.65 | 100.00 | -72.56 | 100.00 | 19.60 | | |

Table 7.10 - Results of Firm 5

Conclusions for the Simulations of Phase II, Group A

- The best overall success ratio and profit performance were grabbed by Firms 3 and 4, obtained from the iterations accomplished with the Consumers situated around and above the median income.
- Firm 5, which offered the highest price, 9.00, achieved virtually 100% of successful iterations with Consumers from Classes 8, 9 and 10, even with the lowest cost, 7.50. This finding demonstrates that, in the absence of competition in that range of cost, there is no point in reducing the profit margin for the mentioned categories of Consumers, if the Firm's objective is improving its participation in the market .
- The model of the market share game imposes limitations to prices and costs to the range [0, 10], mainly because of the design of the fuzzy sets that characterize those variables and the Consumers' reasoning and preferences. Should it be not the case, the results point out to feasibility of greater profits with higher prices, for the upper income Classes of buyers.
- For all the games of the present Group, each with 30000 pairwise encounters, the number of successful iterations varied between 4999 (Games 3 and 4) and 4387 (Game 5). The behavior of the global payoffs for Consumers and Firms evolved in opposite directions, steadily rising and decreasing, for buyers and sellers, respectively. The results are in Table 7.11.

| | Total Payoffs | | | | |
|-----------|---------------|---------|---------|---------|---------|
| | Game 1 | Game 2 | Game 3 | Game 4 | Game 5 |
| Firms | 5585.63 | 4873.28 | 3429.69 | 2369.10 | 219.19 |
| Consumers | 2056.76 | 2529.42 | 2782.86 | 2986.27 | 3117.59 |

Table 7.11 - Total Payoffs received by All Firms and the Population

- The Consumers that belong to Classes 5, 6 and 7 add up to 36.1% of the entire population, and their individual normalized incomes are situated between 0.40 and

0.70, what gives them a good flexibility when considering purchases. This characteristic of the experimental data used in the simulations partially explains why Firm 4 performed well: The combination price/cost offered by this Company reached the Consumers that detain about 51.2% of the total income. In a real market, a fierce competition should be expected in this slice.

- On the other hand, Class 1 has 8% of all members of the population, though possesses only 0.63% of the total wealth. In other words, that Class is a special fraction of the market, which is intensely price oriented and with few demands regarding quality. That is why it was not able to find any satisfactory deal among the presented offers.
- For the set of values experimented and the parameters of the model, the improvement of quality by way of increasing costs was not, in general, productive. In almost all instances the increment in the number of successes was not worth the loss of the absolute amount of profit.

b) Group B, Games 1 - 4: Variable (decreasing) prices, constant costs

In the four games of this round of simulations, the costs for the participating Firms (Firm 0 - Firm 5) have been kept constant, and the prices varied. The criterion for varying prices was not the same for every Firm, and depended on their costs. The inputs used in this series of games are depicted in Table 7.3

The graphs presented next in Figures 7.21(a, b) through 7.26(a, b) show selected success ratios and profits attained by each Firm, as functions of the prices. Except for Firms 0 and 5, which implement cost / price policies aimed at the extreme ranges of incomes, in all other cases the points of maximum success ratio and profit can be clearly identified. Also, as expected, the Classes of Consumers had different behaviors. Recall that sometimes, the penalty imposed to the Firms when a Consumer declines, arbitrarily fixed in 10% of the ratio cost/price, affect their overall outcomes.

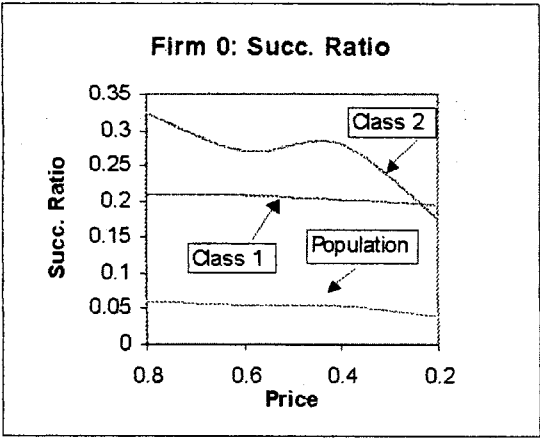


Figure 7.21(a)

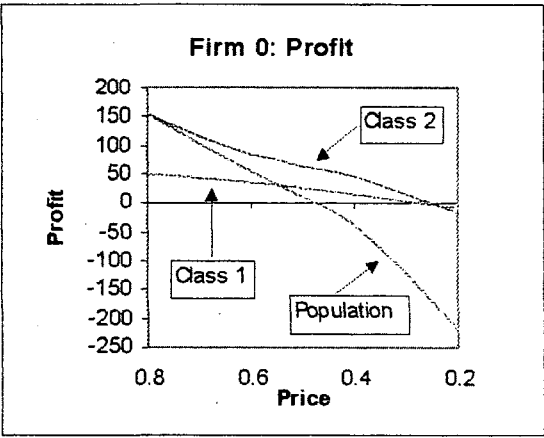


Figure 7.21(b)

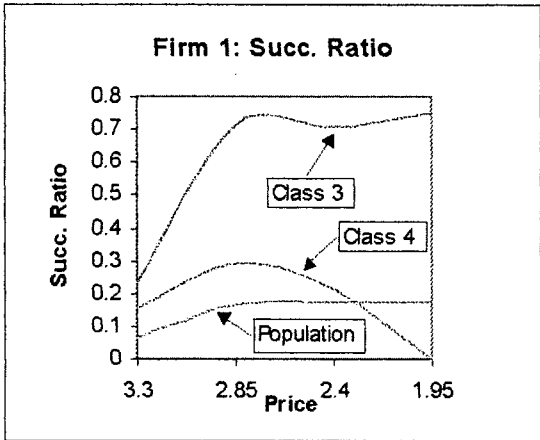


Figure 7.22 (a)

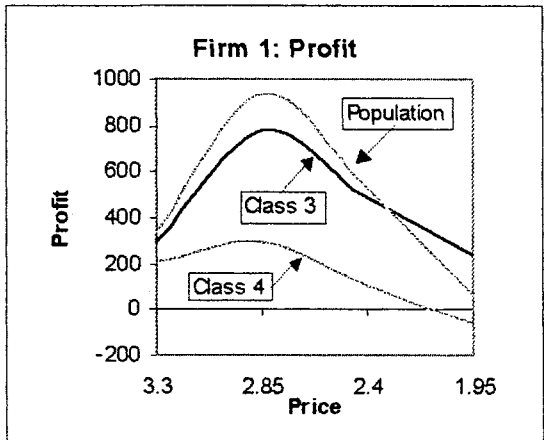


Figure 7.22(b)

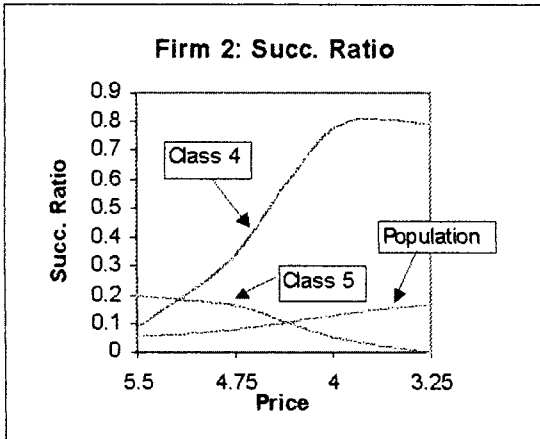


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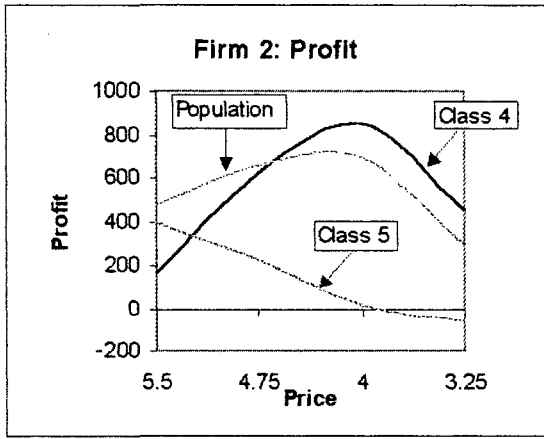


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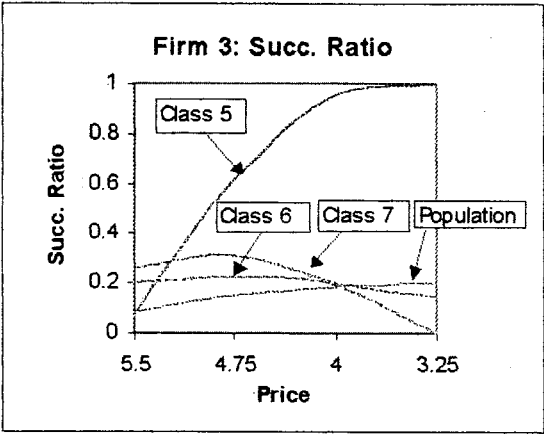


Figure 7.24(a)

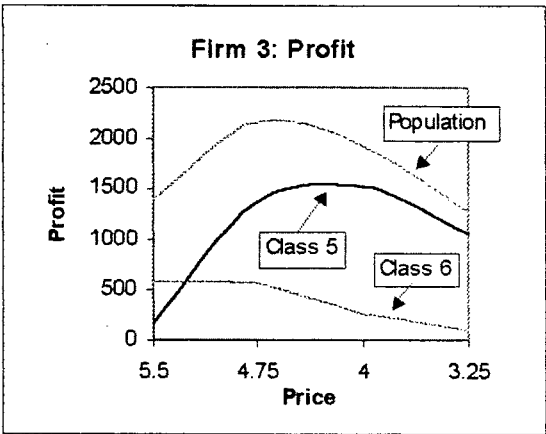


Figure 7.24(b)

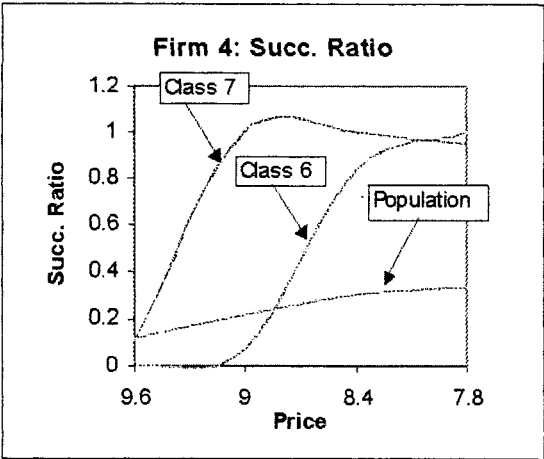


Figure 7.25(a)

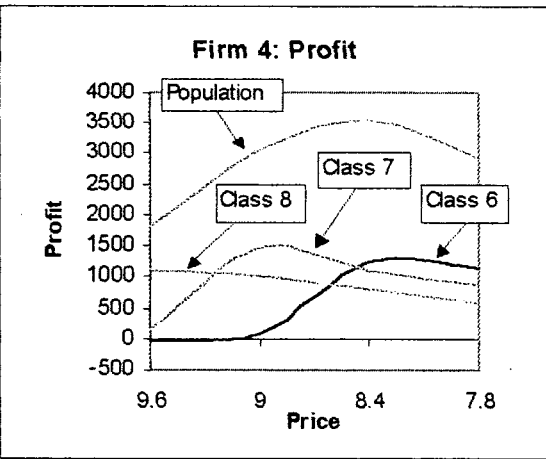


Figure 7.25(b)

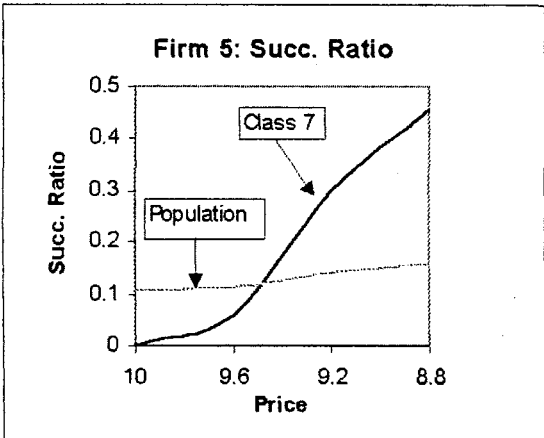


Figure 7.26(a)

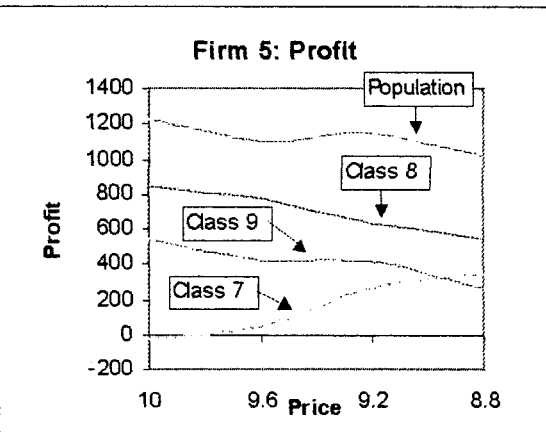


Figure 7.26(b)

The prices for every Firm steadily decreased from Game 1 to 4. From the graphs in the Figures 7.27 and 7.28 it can be seen that a greater *degree of cooperation* on the part of the sellers also entails augmented success ratios, revealing an increased willingness of the Consumers to *cooperate*, too. Nevertheless, regarding the global monetary results for all the Companies as a whole, it appears that the best overall results are achieved when the prices are leveled somewhere around the values employed in Games 2 and 3. This neighborhood, given the current environment, can be interpreted as an *equilibrium* between buyers and sellers, although the benefits bestowed to the former category of players always raise when the relationship price/cost is reduced.

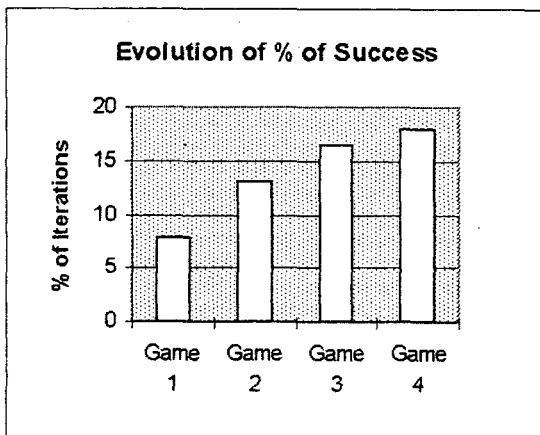


Figure 7.27

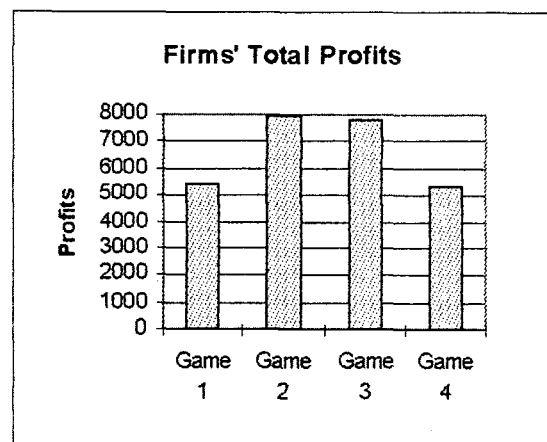


Figure 7.28

Regarding both the volume of sales and the profit, Firm 4 once again revealed itself as the leader of the market. With a fixed cost of MU\$6.00 and prices varying in the range MU\$9.60 - 7.80, it reached all the Classes of Consumers with incomes above the upper-middle level. The satisfaction of the buyers, measured by their totaled payoffs, also had its maximum with Firm 4.

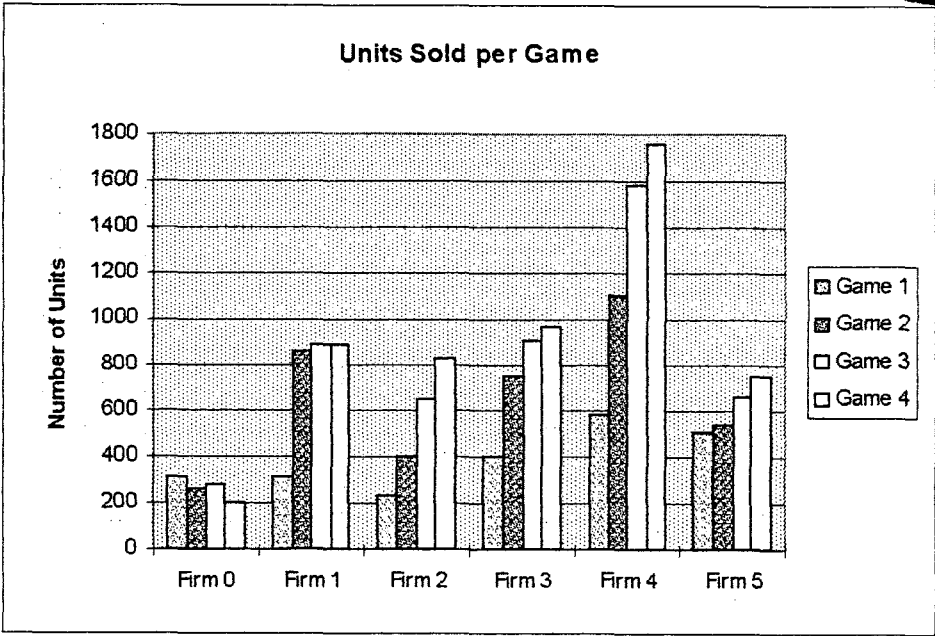


Figure 7.29

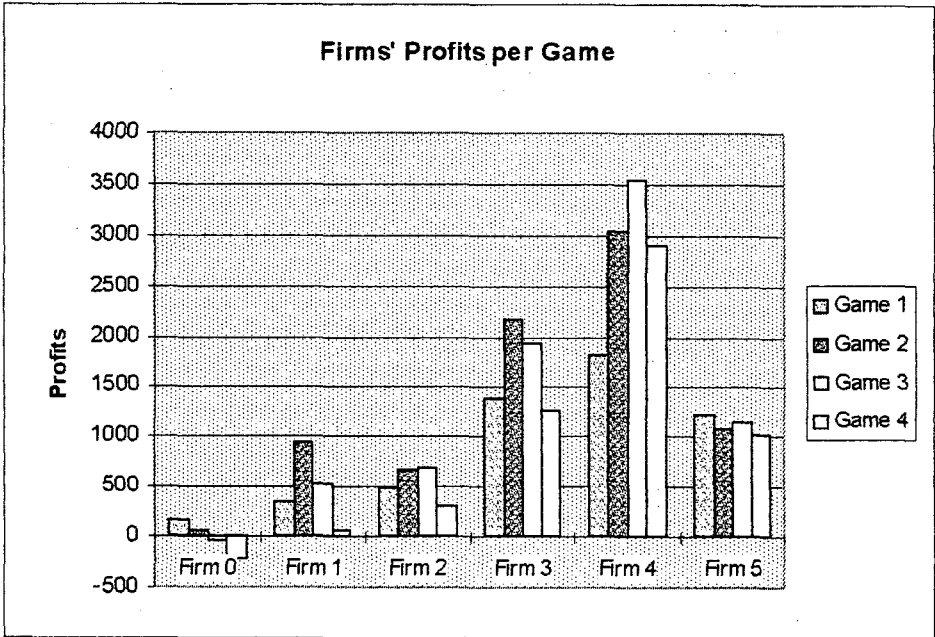


Figure 7.30

Conclusions for the Simulations of Phase II, Group B

- In spite of the reduced costs and prices offered by Firm 0, the Consumers of Class 1 did not accomplish a significant number of buys. The same phenomenon, in a lesser degree, happened with Class 10. This finding suggests a confirmation that the strongest slice of the market pertains to the Consumers situated around the median income, with a bias to the upper end.
- The Consumers of Classes 8, 9 and 10 have been good buyers, with a high ratio of success, close to 100% for Firms 4 and 5. However, given the relatively small number of individuals belonging to those categories, they did not contribute very much to the global results. Those circumstances hint that additional efforts should be made to attract those Classes. This could be made by means of some specifically focused advertising policy, not implemented in the current model.
- In general, the Firms' best results were achieved during Game 3, when the markups had a mode of 60%.
- The items with the lowest cost, offered by Firm 0 and aimed at the less affluent Consumers, supported a large markup (700% in Game 1), and subsequently reducing it (100% in Game 4) did not bring any increased benefits to that supplier. As a matter of fact, the opposite happened, with both the success ratio and profit being reduced with that strategy.
- Class 5 steadily reduced its purchases from Firm 2 (cost = 2.50) when that Firm lowered its prices, but conversely raised the proportion of acquisitions from Firm 3 (cost = 4.0). Classes 6 and 7, though, did not respond favorably to prices below 7.20. This indicates that a higher price can represent a better deal to certain Consumers, even when the quality associated with the item remains untouched.
- For Group B, the number of successful iterations varied between 2342 (Game 1) and 5388 (Game 4). The evolution of the Firms' payoffs behaved differently from the Games of Group A— the slope of the profit curve changed its signal within the

considered interval—, but the Consumers’ gains increased consistently with the reduction of markups. respectively. The results are in Table 7.12.

| | Total Payoffs | | | |
|-----------|---------------|--------|--------|--------|
| | Game 1 | Game 2 | Game 3 | Game 4 |
| Firms | 5395 | 7955 | 7807 | 5342 |
| Consumers | 443 | 1025 | 1855 | 2708 |

Table 7.12- Total Payoffs received by All Firms and the Population

7.3.4 - The Simulations of Phase III

In each of the six games played in this Phase, six Firms competed in the same market segment, with an equivalent cost and differentiated prices. In the analysis that follows, the results obtained from Game 1 contain some additional detail, since the participating Companies aimed their items at the Consumers of lower income, specially Class 1, which has demonstrated strong limitations in its purchase power in the previous experiments.

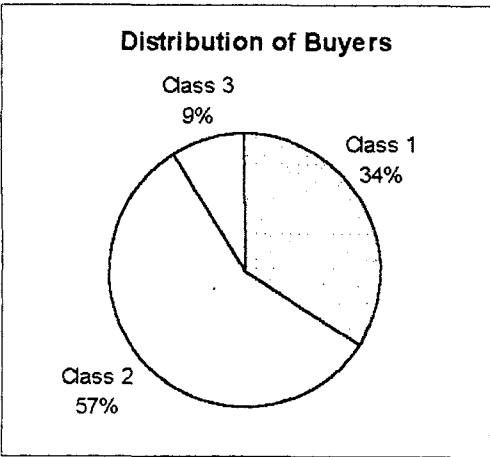


Figure 7.31 - Customers of Game 1

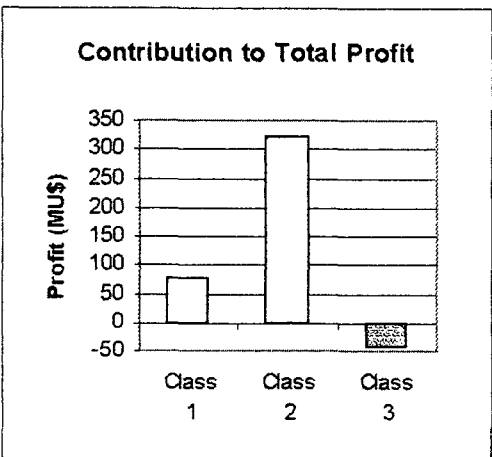


Figure 7.32 - Firms’ Profits from Customers

In Game 1, where the cost was constant and equal to 2.50, only Classes 1, 2 and 3 turned out as actual customers. All other Classes did not accomplish any purchase. Class 1,

though contributing with 34% of all successful iterations—Figure 7.31, did not supply the greatest profit to the Firms, not even to Firm 1, which had the lowest markup (60%). Class 3, which initially bought from the Firms, on account of the introductory optimistic decision criterion of the model, quickly withdraw from the market, as negative payoffs accumulated. The data shown in Figures 7.32 regard only the outcomes that derived from the iterations with those three Classes, hence excluding the negative payoffs yielded by non-successful iterations with the other Consumers.

Firm 0 contrasted the larger slice of the market (35%) with the lowest profit, 3% of the total. Firms 4 and 5, with markups of 300% and 360%, respectively, attained the best monetary results with only a modest market share, as illustrated by Figures 7.33 and 7.34. Firm 3 performed worse than Firm 2 and 4, in spite of the fact that its markup (240%) lies between those of the former Companies. An explanation for this occurrence concerns the preference functions of the consumers, since the combination of Firm 3's cost and price was possibly disadvantageous in presence of the competition from both sides.

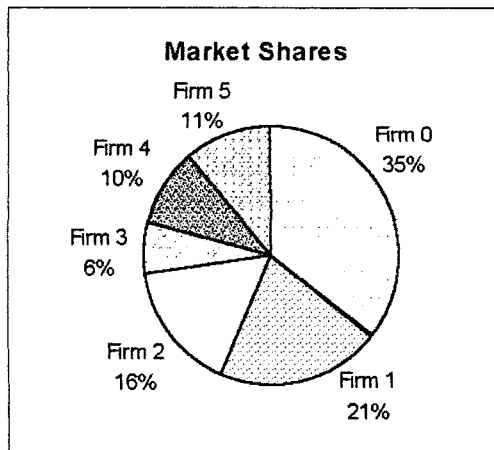


Figure 7.33 - Distribution of Successful Iterations

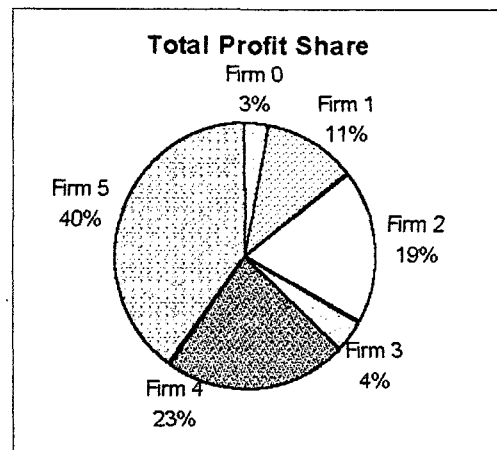


Figure 7.34 - Distribution of Profits

The graphs that follow illustrate, for each Game, comparisons between the total number of successful iterations and the monetary results achieved, as functions of the current markup. Now the profit has been computed taking into account the gains and losses obtained with the iterations with *all Classes*, whether successful or not.

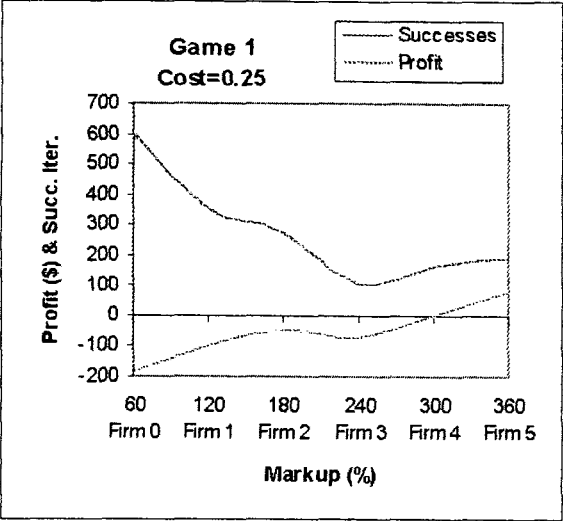


Figure 7.35

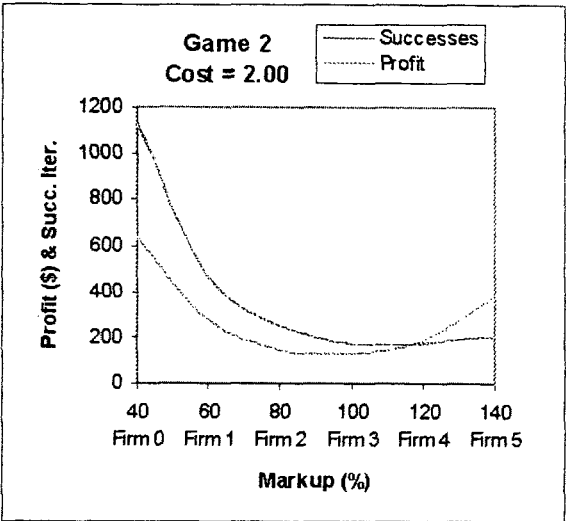


Figure 7.36

In terms of the population as a whole, the deals offered by the Firms in Game 1 proved to yield very low profits. In fact, only with very large markups (360%) a positive gain appeared. This rather negative result is due, mostly, to the refusal of the wealthier Consumers. Even with the lowest markup, the percentage of success achieved in Game 1 was low, around 5%. As to Game 2, the best overall results also took place when the markup was the lowest, in this case 40%. Here, the proportion of success almost doubled, and positive gains were always present for any markup considered.

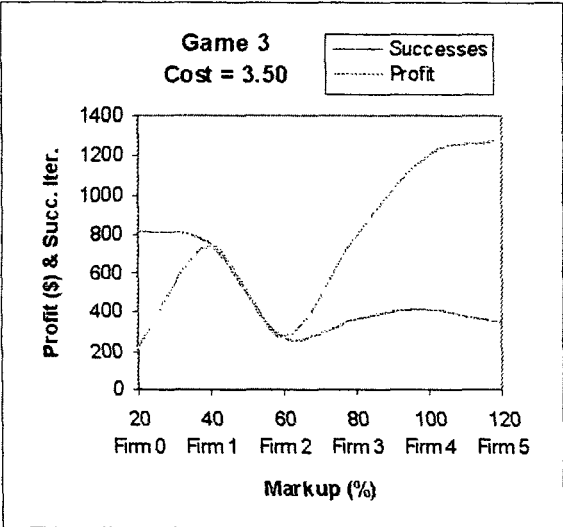


Figure 7.37

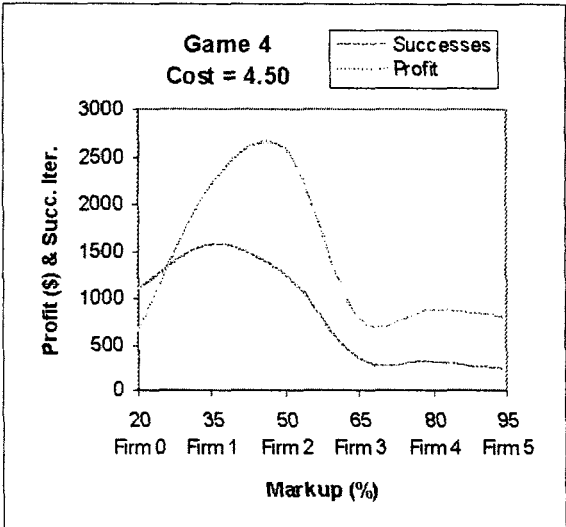


Figure 7.38

In Game 3 the Consumers, taken collectively, appear to show little variation in the rates of buys for the markups 20% - 40% and 80% - 120%. Regarding the markups, the total profit had a local maximum at 40%, but the global optimum is around 100%, which also coincides with a local maximum in the number of successes. Conversely, Game 4 presents a clear point of maximum profit, approximately associated with the greatest number of success, with a margin of about 45%.

The best policies for unitary profit margins with respect to costs varying from 5.50 to 6.50 seems to lie in the vicinity of markups between 40% - 50%, as demonstrated by the graphs in Figures 7.39 and 7.40. Although the wealthier Consumers, to whom the simulations of Game 6 were directed, have a low sensibility to price, the absence of competitors operating in a different cost/quality range caused them to reduce their buying rate when the markup was higher. Nevertheless, the profit increased, showing that for that category of Consumers the attainment of a maximum success rate is not strongly correlated with monetary results.

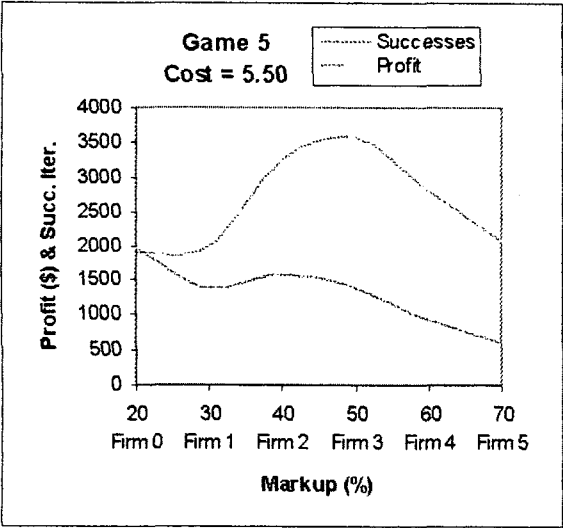


Figure 7.39

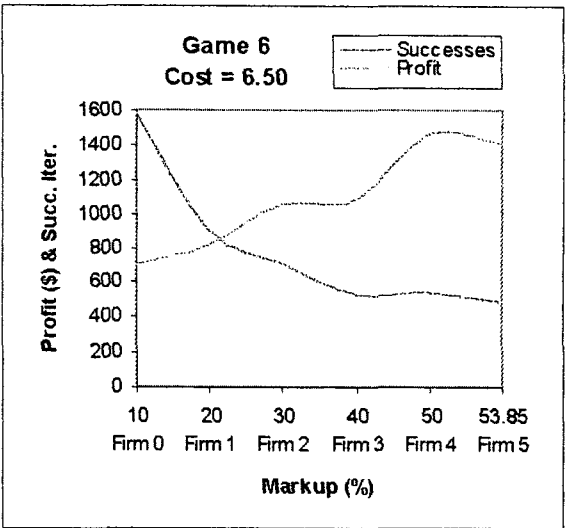


Figure 7.40

Conclusions for the Simulations of Phase III

- The best result in terms of net profits were achieved in Game 5 (cost = 5.50) by Firm 3 with the price \$8.25, which received a total of more than MU\$3500. The corresponding success ratio was about 4%. All other sellers in that game also performed well, with a minimum profit of MU\$2000.
- The highest success ratio in this group of games also occurred in Game 5, and it has been detained by Firm 0 ($\approx 7\%$), but it did not coincide with the greatest profit.
- The Consumers of Class 1 contributed with very little to the Firms' gains, when exposed to the same probability of iterating as the other Classes. One line of action foreseen to handle this slice of the market is (i): Work with low costs (below \$0.50) and relatively large markups (300% and beyond); (ii) Increase the volume of sales by raising advertising expenses *specifically aimed* to that category, thus avoiding unproductive attempts of conquering other Classes of Consumers.
- Though informative, the results obtained from this Phase of experiments cannot be straightforwardly transferred to other situations of competition because only one alternative of cost/quality was present in each game. It must be recalled that both the Consumers' evaluation functions and decision thresholds are quite context-dependent.
- In general terms, the best buyers were the Consumers from the middle and upper-middle Classes, which are much more flexible concerning their spectra of favorable deals.
- The relation that yielded the most attractive results for the Firms is located around the cost \approx \$5.00 and markup of 40 - 50%.

7.3.5 - The Simulations of Phase IV

In this Phase, the effect of varying the advertising budgets of the Firms has been investigated. For this purpose, two games were played, with four Firms each, competing in the same range of cost / quality, but with different prices, shown in Table 7.13.

| | Cost | Price | Markup (%) |
|--------|------|-------|------------|
| Firm 0 | 0.25 | 0.40 | 60 |
| Firm 1 | 0.25 | 0.55 | 120 |
| Firm 2 | 0.25 | 0.70 | 180 |
| Firm 3 | 0.25 | 0.85 | 240 |

Table 7.13 - Phase IV: Prices and Costs for Games 1 & 2

In the simulations of Game 1, all four Firms had no differentiation regarding their advertising budget and, consequently, the probability of an encounter with any Consumer was the same, namely, 25%. In Game 2, the Firms' advertising expenses were different, thus altering the frequency of their iterations. However, considering that the level of cost and prices were aimed at the buyers with small incomes, only the Consumers of Classes 1 and 2 will be taken into account in the analysis that follows, since the other Classes either performed an insignificant number of purchases (Class 3), or simply refused to buy at all.

The data obtained in the Game 1, Group B, Phase III will be used as a basis for establishing a criteria for the differentiation of the advertising budgets, as well as the prices and costs of the four participating Firms, which correspond to those of the equally numbered Companies in the mentioned runs of simulations. The parameters for the advertising expenses of Game 2 of the current Phase have been made **inversely proportional** to the profit share previously obtained. The total amount expended was arbitrated as 20% of the total absolute profit of Game 1, Group B, Phase III. The rightmost column of Table 7.14 depicts the values that will be added to the gross profit achieved in the iterations.

| | Absolute Profit ⁶ | Profit Share | Virtual Advertising Budget for Game 1 ⁷ | Advertising Parameters for Game 2 | Virtual Advertising Budget for Game 2 ⁸ | Effective Advertising Budget for Game 2 ⁹ |
|--------|------------------------------|--------------|--|-----------------------------------|--|--|
| Firm 0 | 57.73 | 0.2191 | 13.18 | 0.2518 | 13.2719 | 0.0919 |
| Firm 1 | 75.03 | 0.2847 | 13.18 | 0.1937 | 10.2095 | -2.9705 |
| Firm 2 | 94.51 | 0.3586 | 13.18 | 0.1538 | 8.1065 | -5.0735 |
| Firm 3 | 36.27 | 0.1376 | 13.18 | 0.4007 | 21.1201 | 7.9401 |
| Total | 263.54 | 1 | 52.7 | 1 | 52.7 | 0.0 |

Table 7.14 - Determination of the Differentiated Advertising Budget for Game 2, Phase IV

From Equation 6.8 of Chapter 6, section 6.4.4 II, the advertising budget to be assigned by each Company was given by $A_{ig} = a_{ig} \times 10 \times I$, where a_{ig} is a fraction of the maximum possible profit, 10, and I is the expected number of iterations in the current cycle of the game. In the present instance, since only Classes 1 and 2 of Consumers are being examined, I will be computed as 0.217×20000 , because the two mentioned Classes account for 21.7% of the entire population. In this manner, the values termed *virtual advertising budget* shall coincide with A_{ig} , with the variable a_{ig} assuming the values

⁶ The absolute profit and the profit share were extracted from Game 1, Group B, Phase 3 of the simulations.
⁷ Calculated as 20% of the total absolute profit \times equivalent advertising parameters (0.25).
⁸ Calculated as 20% of the total absolute profit \times differentiated advertising parameters.
⁹ Calculated as the difference between the virtual advertising budgets of Game 2 and Game 1.

0.003058, 0.002352, 0.001868 and 0.004866, respectively for Firms 0, 1, 2 and 3¹⁰. It must be noted that the objective of the simulations of this Phase has been to evaluate the influence of contrasted advertising budgets, while all other variables present in the two games under consideration are kept unchanged. Thus, only the *marginal* advertising expenses—the difference between both *virtual* budgets, shall be taken into account to compute the final results, *videlicet*, the last column of Table 7.14.

The results yielded by the simulation program for Game 1 are summarized in the sequence.

I. Tables 7.15(a) - (e): Game 1 - Equivalent Advertising Budgets

| Firm 0 | | | | | |
|--------|----------------|----------------|-------------|-------------|--------------|
| Class | No. Iterations | No. Abs. Succ. | Succ. Ratio | Abs. Profit | Profit Ratio |
| 1 | 441 | 441 | 1.00 | 66.150 | 1.135 |
| 2 | 721 | 175 | 0.24 | -7.875 | -0.135 |
| 1 & 2 | 1162 | 616 | 0.53 | 58.275 | 1.0 |

Table 7.15(a) - Phase IV, Game 1: Simulation Partial Results for Firm 0

| Firm 1 | | | | | |
|--------|----------------|----------------|-------------|-------------|--------------|
| Class | No. Iterations | No. Abs. Succ. | Succ. Ratio | Abs. Profit | Profit Ratio |
| 1 | 424 | 80 | 0.19 | 8.3635 | 0.166 |
| 2 | 681 | 211 | 0.31 | 41.9361 | 0.834 |
| 1 & 2 | 1105 | 291 | 0.26 | 50.2996 | 1.0 |

Table 7.15(b) - Phase IV, Game 1: Simulation Partial Results for Firm 1

¹⁰ Although the values of a_g appear to be quite small, it should be remembered that they regard a fraction of the utmost plausible profit to be attained in a cycle, unrealistic in practice, and serving exclusively as a reference.

| Firm 2 | | | | | |
|--------|----------------|----------------|-------------|-------------|--------------|
| Class | No. Iterations | No. Abs. Succ. | Succ. Ratio | Abs. Profit | Profit Ratio |
| 1 | 401 | 78 | 0.19 | 23.564 | 0.440 |
| 2 | 682 | 112 | 0.16 | 30.043 | 0.560 |
| 1 & 2 | 1083 | 190 | 0.18 | 53.607 | 1.0 |

Table 7.15(c) - Phase IV, Game 1: Simulation Partial Results for Firm 2

| Firm 3 | | | | | |
|--------|----------------|----------------|-------------|-------------|--------------|
| Class | No. Iterations | No. Abs. Succ. | Succ. Ratio | Abs. Profit | Profit Ratio |
| 1 | 394 | 25 | 0.06 | 4.147 | 0.074 |
| 2 | 630 | 112 | 0.18 | 51.965 | 0.926 |
| 1 & 2 | 1024 | 137 | 0.13 | 56.112 | 1.0 |

Table 7.15(d) - Phase IV, Game 1: Simulation Partial Results for Firm 3

| All Firms | | | | | |
|-----------|----------------|----------------|-------------|-------------|--------------|
| Class | No. Iterations | No. Abs. Succ. | Succ. Ratio | Abs. Profit | Profit Ratio |
| 1 | 1660 | 624 | 0.38 | 102.226 | 0.468 |
| 2 | 2714 | 610 | 0.22 | 116.070 | 0.532 |
| 1 & 2 | 4374 | 1234 | 0.28 | 218.295 | 1.0 |

Table 7.15(e) - Phase IV, Game 1: Simulation Partial Results for All Firms

II. Tables 7.16(a) - (e): Game 2 - Differentiated Advertising Budgets

| Firm 0 | | | | | |
|--------|----------------|----------------|-------------|-------------|--------------|
| Class | No. Iterations | No. Abs. Succ. | Succ. Ratio | Abs. Profit | Profit Ratio |
| 1 | 429 | 429 | 1.000 | 64.350 | 1.069 |
| 2 | 733 | 196 | 0.267 | -4.163 | -0.069 |
| 1 & 2 | 1162 | 625 | 0.538 | 60.188 | 1.0 |

Table 7.16(a) - Phase IV, Game 2: Simulation Partial Results for Firm 0

| Firm 1 | | | | | |
|--------|----------------|----------------|-------------|-------------|--------------|
| Class | No. Iterations | No. Abs. Succ. | Succ. Ratio | Abs. Profit | Profit Ratio |
| 1 | 298 | 79 | 0.265 | 13.746 | 0.237 |
| 2 | 571 | 203 | 0.356 | 44.173 | 0.763 |
| 1 & 2 | 869 | 282 | 0.325 | 57.918 | 1.0 |

Table 7.16(b) - Phase IV, Game 2: Simulation Partial Results for Firm 1

| Firm 2 | | | | | |
|--------|----------------|----------------|-------------|-------------|--------------|
| Class | No. Iterations | No. Abs. Succ. | Succ. Ratio | Abs. Profit | Profit Ratio |
| 1 | 323 | 76 | 0.235 | 25.378 | 0.440 |
| 2 | 469 | 101 | 0.215 | 32.307 | 0.560 |
| 1 & 2 | 792 | 177 | 0.223 | 57.686 | 1.0 |

Table 7.16(c) - Phase IV, Game 2: Simulation Partial Results for Firm 2

| Firm 3 | | | | | |
|--------|----------------|----------------|-------------|-------------|--------------|
| Class | No. Iterations | No. Abs. Succ. | Succ. Ratio | Abs. Profit | Profit Ratio |
| 1 | 527 | 22 | 0.042 | -1.653 | -0.038 |
| 2 | 936 | 115 | 0.123 | 44.853 | 1.038 |
| 1 & 2 | 1463 | 137 | 0.094 | 43.200 | 1.0 |

Table 7.16(d) - Phase IV, Game 2: Simulation Partial Results for Firm 3

| All Firms | | | | | |
|-----------|----------------|----------------|-------------|-------------|--------------|
| Class | No. Iterations | No. Abs. Succ. | Succ. Ratio | Abs. Profit | Profit Ratio |
| 1 | 1577 | 606 | 0.384 | 101.822 | 0.465 |
| 2 | 2709 | 615 | 0.227 | 117.171 | 0.535 |
| 1 & 2 | 4286 | 1221 | 0.285 | 218.993 | 1.0 |

Table 7.16(e) - Phase IV, Game 2: Simulation Partial Results for All Firms

Observing the results from Game 2, it can be seen that the number of iterations in which Firms 1 and 2 took part in fact decreased in comparison to Game 1, a natural consequence of the bias imposed by random selection process. However, the accumulated profit accumulated along the iterations had an opposite behavior, slightly increasing, even without the reduction due to smaller advertising expenses. Why this? The main reason for that occurrence is the presence of Firm 0, which offered a better deal to the Consumers of Classes 1 and 2 by means of a significant smaller markup over the commonly established cost. Because of the competition presented by Firm 0, which always maintained 100% of buyers, the success ratio of the other Firms steadily decreased as the game proceeded, as illustrated by Figure 7.41.

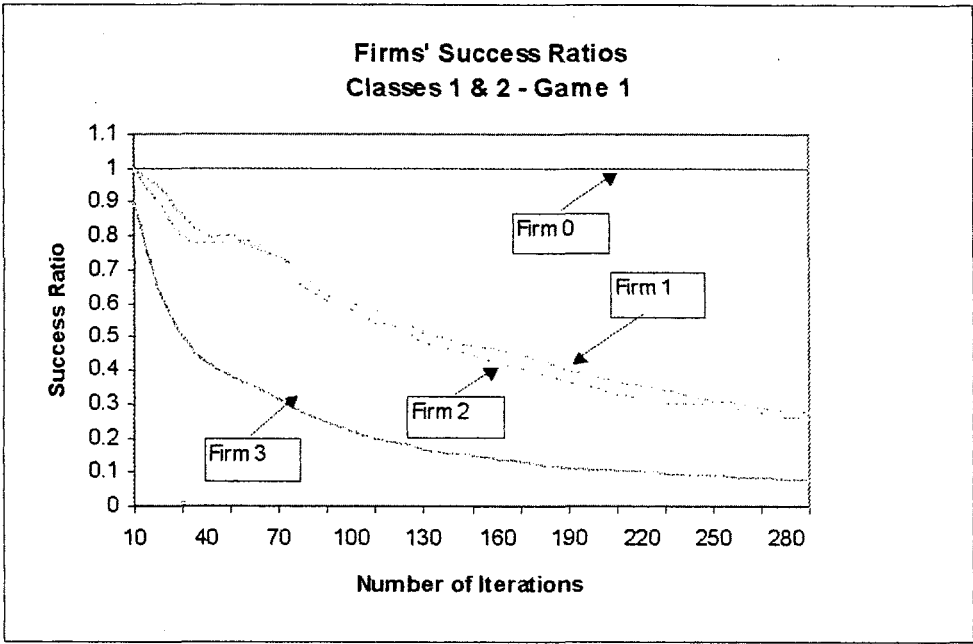


Figure 7.41 - Conjoint Success Ratio for Classes 1 & 2, Phase IV, Game 1

A comparable event took place with Firm 1, 2 and 3's accumulated payoffs, whose results regarding Class 1 are depicted in Figure 7.42. For Class 2 the circumstances were similar.

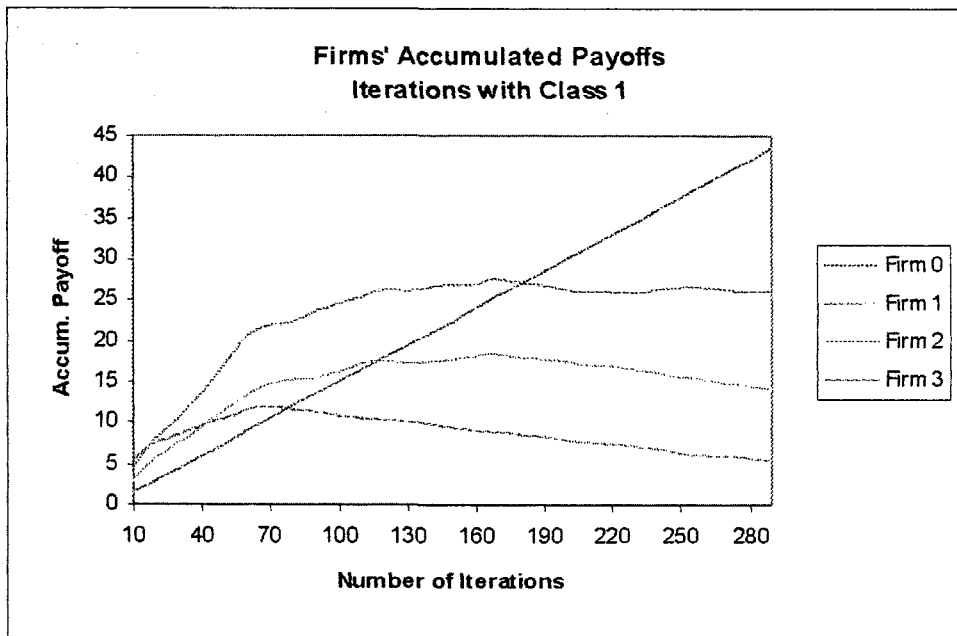


Figure 7.42 - Firms' Accumulated Payoffs with Class 1, Phase IV, Game 1.

According to the rules of the game, non-accomplished purchases mean sure losses—the penalty for the Consumer's *defection*. In this manner, there is no point in trying to increase the number of iterations if they necessarily translate into negative gains. This is why Firms 1 and 2 ended up better off in Game 2, in spite the fact that their frequency of iterations had been reduced, comparatively to Game 1.

On the other hand, an opposite phenomenon happened to Firm 3, which boosted its number of encounters in Game 2 for no avail. Recall that Firm 3 had the highest markup.

The total gains obtained by the Firm 0 and 3 in Game 1 with Classes 1 and 2 taken aggregated do not differ very much—Figure 7.43—, basically because of the varied decision rules of the Consumers are compensated by larger unitary profits. Class 1 was the greatest contributor to the revenues obtained by the Firm with the lower price, and the opposite occurred with Class 2.

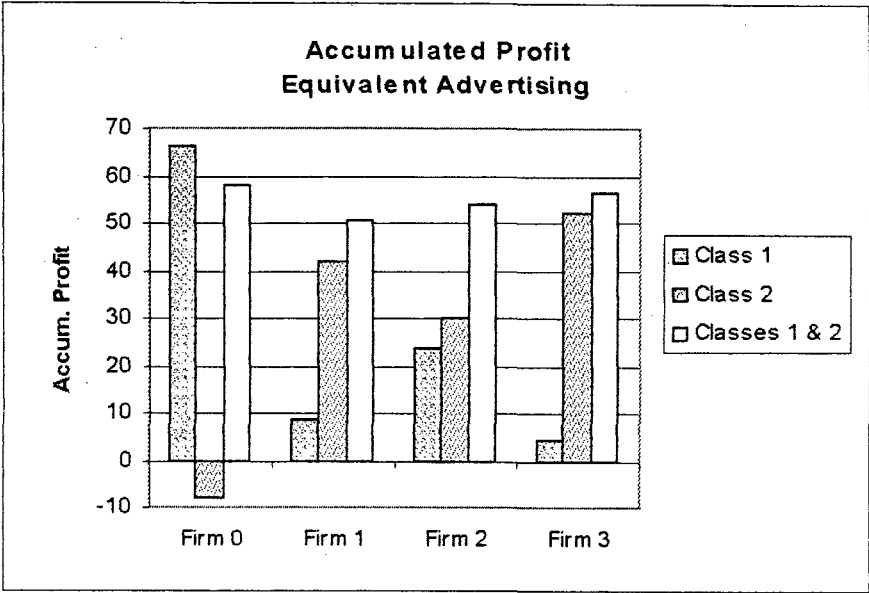


Figure 7.43 - Total Profit in Game 1 (without advertising expenses)

When the Firms act on their probability of being chosen for an iteration by employing diverse advertising budgets, it can be seen that, while Firm 0 practically had no alterations in its profit, Firms 1 and 2 jumped to more favorable results, and Firm 3, to a worse one.

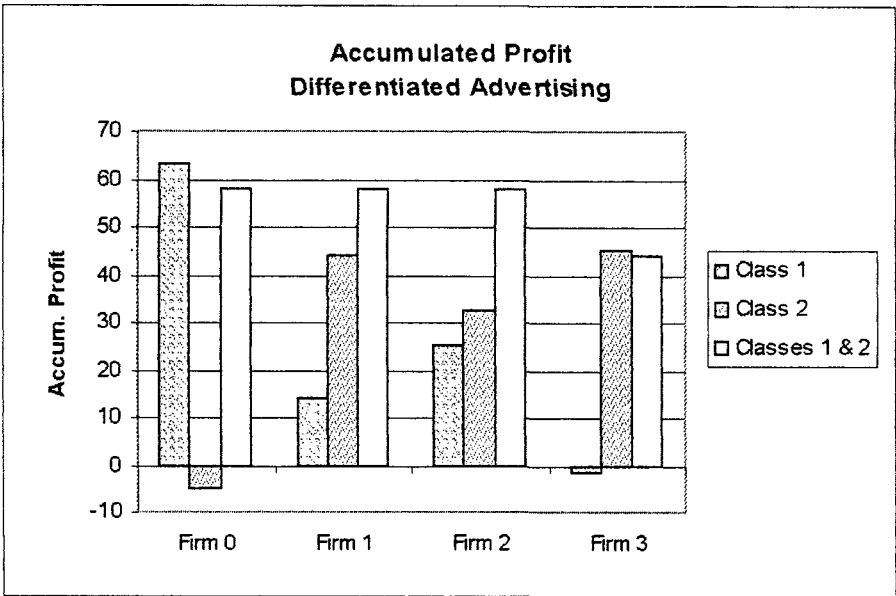


Figure 7.44 - Total Profit in Game 2 (without advertising expenses)

Now, when the values of the *effective advertising budget*— Table 7.14, rightmost column— are added to the outcomes shown in Figure 7.44, there will be an intensification of the foregoing comments.

Figures 7.45 and 7.46 summarize the general results concerning the profit and the total number of iterations and successes obtained in Phase IV.

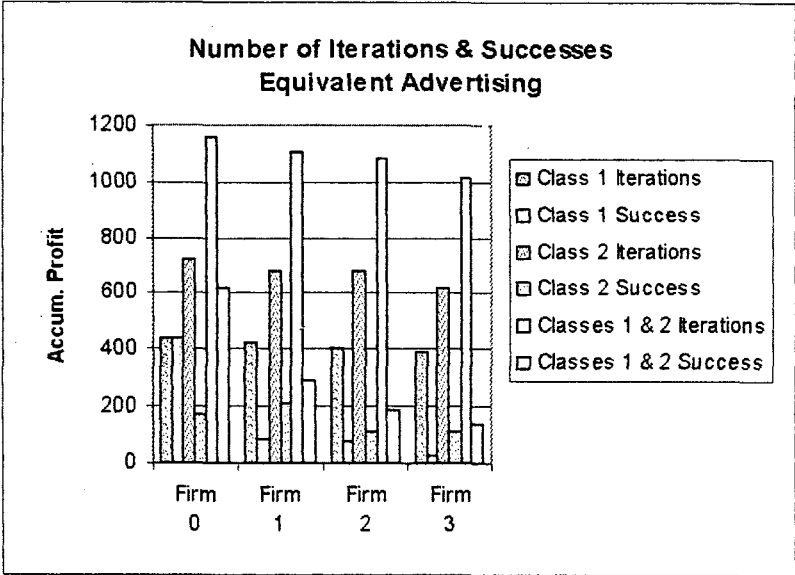


Figure 7.45 - Game 1, Phase IV

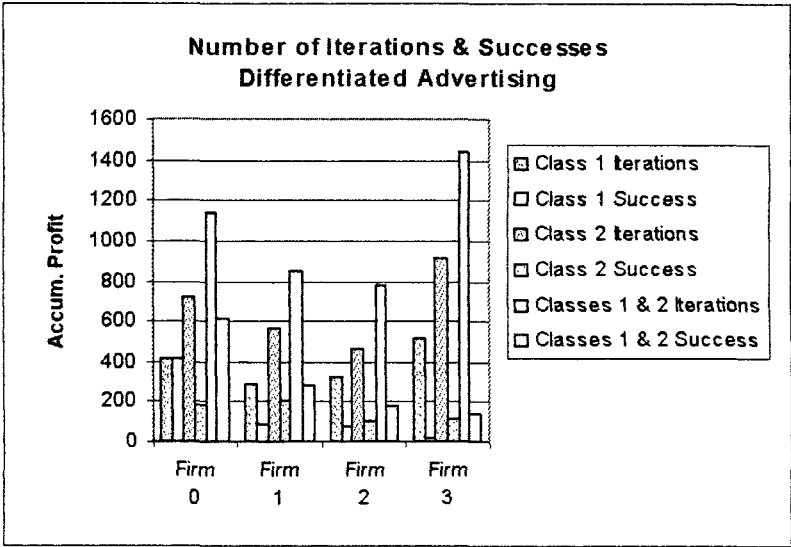


Figure 7.46 - Game 2, Phase IV

Conclusions for the Simulations of Phase IV

- In this Phase, two Games were played: Game 1, with no differentiation in the Firms' advertising budgets, hence with the same expected number of iterations for each Firm, and Game 2, where the Companies had distinct probabilities of performing iterations, due to particular publicity costs.
- When the sellers compete for the same spectrum of buyers, the results indicated that the Firms which offer the items of same cost/ quality at higher prices soon lose their ability to accomplish sales, because the Consumers rapidly prefer the deals where they can find better advantage. This outcome, though intuitively obvious, indicates that the model is adequate in terms of mirroring practical situations.
- Increasing advertising expenses in the presence of more efficient competition has demonstrated to be useless; Quite on the contrary, because, since there is a *penalty* imposed to the Firms when the Consumers are called to iterate and refuse to buy, the accumulation of negative payoffs due to sequential *refusals* pulls the final profit to a worse position.
- Firm 0, which had the best overall performance in Game 1, maintained its rank in Game 2. On the other hand, Firms 1 and 2, which had a smaller number of iterations, had greater profits, on account of the foregoing cited reasons.

7.4 - Final Remarks

In this Chapter, the problem of analyzing several instances of a market competition has been simulated with the *One-sided Fuzzy Iterated Prisoner's Dilemma*—1S-FIPD. Regarding the players embodied by the Firms, the flavor of the classic Prisoner's Dilemma is actually present. Although the motivation of using this paradigm in this kind of problem has already been discussed in Chapter 6, it is opportune to mention that the basic dilemma of the traditional IPD is also extant in the workings of the 1S-FIPD model of a market share game. Some significant similarities regarding those two approaches are:

- i. The Firms— in the 1S-FIPD, the category of players that is actually involved in a PD— can either cooperate or defect, in a gradual manner, by establishing diverse levels of markups;
- ii. Defection pays, until a certain extent, since the decision functions used by the Consumers tend to initiate the game in a cooperative mood, which is quickly and constantly updated.
- iii. However, the Consumers are reactive decision makers, and *defection* is soon answered with defection, which implies in losses to the sellers. The Consumers' *defection* is symbolized by the act of not buying from the non-cooperative Firms;
- iv. In the long run, if well adjusted, cooperation is worth, yielding positive though smaller payoffs to both sides involved.

Chapter 8

“...the individual pursuit of a large share of the pie often sabotages efficiency.”

—J. McMillan, in *Games, Strategies and Managers*, 1992.

Conclusions

8.1 - Summary of the Dissertation

The vast majority of studies and applications of the IPD regard a 2×2 game, with two players and two mutually exclusive strategies, COOPERATE and DEFECT, with the participants usually formulating their decisions by means of some appraisal of the foregoing sequence of moves.

In this Dissertation, the IPD was explored with some alternative and new approaches, basically:

- Admittance of *gradations* in the strategies, between the extreme points C and D;
- Formulation of decision systems for the players that incorporate other variables besides the history of previous encounters;
- Introduction of fuzzy logic and fuzzy expert systems to ascribe a qualitative reasoning method for the players—*everything is a matter of degree*¹;

¹ This quotation is borrowed from Dr. A. Kandel, who used it in a course on fuzzy systems attended by the author of this dissertation in the second semester of 1994 at the University of South Florida.

- Employment of a *one-sided* Prisoner's Dilemma to model a market share game;

The Dissertation is informally divided in 2 parts. Chapters 1 through 4 contain the introduction and a review of the fundamental theoretical aspects of Game Theory, the Prisoner's Dilemma Game and Fuzzy Sets. In Chapter 4, a contribution was also made regarding the aggregation of decision criteria using the *fuzzy integral*. A modification to the original approach of this method was studied and implemented, which comprises the inclusion of an *increment* Δ^+ and a *decrement* Δ^- in the formulation of $\int_F \alpha \partial \phi$. Chapters 5 through 8 comprise exploratory models of the Fuzzy Iterated Prisoner's Dilemma—FIPD, and a practical application of the FIPD using the one-sided version of the game with its simulations results, and the conclusions.

One significant portion of the work regarded the creation of two specially written object-oriented C++ codes to run the simulations concerning the models presented in Chapters 5 and 6, since no commercially available package fulfilled the specifications to perform the required tasks.

The achievement of a greater insight of the workings of the PD paradigm has been the major goal of the present research, and the most important acquired outcomes are summarized in the sequence

8.2 - Synopsis of the Results

8.2.1 - The FIPD Tournaments (Chapter 5)

The tournaments consisted in confronting the fuzzy players, differentiated by their respective rules, against themselves and simultaneously against other traditionally successful strategists in earlier contests, TFT and PAVLOV. Those established rules maintained their usual dichotomic character regarding the criteria of assessment of previous moves and implementation of decisions. The results that have been obtained from the limited set of

simulations performed showed a quite erratic pattern concerning efficiency in accumulating payoffs. In the first phase of the iterations, PAVLOV, which uses the simple rule *win-stay, lose-change*, was the overall winner in most instances. In this stage, TFT did not thrive. As to the fuzzy players, a noticeable feature was the prevalence of *defection-biased* strategists among the best performers, indicated by the presence of the digits 4, 5, 6, 7 in their fuzzy classification numbers². In short, the predominantly *nice* and *forgiving* fuzzy strategists did not succeed in the competitive environment.

Nevertheless, the realization of the subsequent phases of the tournament, from which the poorest performers have been excluded, revealed a change in the results. From the second phase on, PAVLOV abandoned its success and TFT began to shine, only eventually defeated by the defection-biased fuzzy rules. On the other hand, fuzzy-TFT did not leave the ground, and the binary TFT ended up as the final winner of the tournaments.

Three relevant aspects concerning the simulations are worth mentioning, though:

- The first is the fact that PAVLOV and TFT (binary and fuzzy versions) were always picked to participate in every phase of the simulations, in contrast with the exclusion criterion applied to the other players. This criterion provided an advantage to those participants, which had renewed opportunities to recover from precedent comedowns.
- The second topic refers to the influence of the probabilistic characteristic of sequential assignment of pairs of players to being confronted. Because the fuzzy expert systems take into consideration a set of data which is derived from the history of the game, the resulting decisions that were adopted along the tournament phases were strongly influenced by the particular succession of pairwise encounters. Although the cited overall recipe for good performance appears to have been confirmed in hundreds of thousands of iterations, there still remains a speculation whether a definite and unique stable configuration might really arise from the equally many possible starting arrangements. This question induces a conjecture about the an inherent unpredictable

² The digit 5 stands for widespread defection in every circumstance, and 4, 6, 7 correspond to defection in two out of three resolution clauses.

trait of complex and dynamic social (and other) systems, where some small fortuitous changes can make a big difference.

- The third point consigns the frequency distribution of the types of players. In the model there were 512 distinct fuzzy categories, and three other special participants. However, each type counted with *only one* representative in all matches. An ecologically designed tournament comes immediately to mind, where the most apt strategies proliferate and the worst become extinct. Nonetheless, even in this circumstance, it appears that the considerations about the intrinsic unpredictability of the resulting equilibrium still endure.

8.2.2 - The One-Sided FIPD—1S-FIPD—and its application to a Market Share Game (Chapters 6 and 7)

Chapters 6 can be considered the nucleus of the Dissertation. A model based on the 1S-FIPD has been implemented and extensively detailed. Along the text, several problems concerning the relationship between the conflict of interest posed by a market with Firms and Consumers, the IPD and the use of methods based in fuzzy set theory were approached and successfully formulated.

In Chapter 7, with the aid of an object-oriented C++ program, the 1S-FIPD has been simulated under a number of different aspects, which were divided in four phases, summarized below.

- Phase I: A general tournament with six Firms operating at diverse market segments regarding the quality and cost of items. The game was divided in five *cycles*. From a cycle to the next, half of the Companies made alterations in their prices, while the costs were kept constant for all Firms. Among the diverse conclusions achieved, it was found that the most successful Company was number 4, which operated with fixed cost and prices throughout the cycles, and aimed them at the upper-middle Classes of Consumers.
- Phase II: Here, two groups of Games were simulated. In Group A, the objective was observing the effect of conjoint of diminishing costs and fixed prices on the

performance of the Firms. In Group B, the opposite policy was adopted, with constant costs and decreasing prices. Following a characteristic already found in Phase 1, the Consumers from Class 1— with the smallest incomes— were quite reticent in accomplishing purchases. In fact, there were no buyers from that category in Group A, at a price of MU\$1.00. In Group B, their general contribution to the Firms' profits was insignificant. The best contributors to the Companies' profits were the Classes 5, 6 and 7, which, aside being more numerous, also had more flexibility in their decision criteria, being less specific in their preferences and buying from more sellers.

Another interesting finding was that the more efficient markup in terms of obtaining a maximum profit was between 40% and 60%.

- Phase III: This stage consisted of six games, each with six Firms competing together in the same slice of the global market. The costs were fixed in each game, and the sellers offered their items with different markups.

The best result in terms of net profits were achieved in Game 5, when the cost was 5.50 and the price, \$8.25, which corresponds to a markup of 50%, thus confirming the same optimum range found in Phase II. The respective success ratio was about 4%. The highest success ratio in Phase III occurred also in Game 5, and it has been detained by Firm 0 ($\approx 7\%$), but it did not coincide with the greatest profit. This result evinces that the 1S-FIPD model is appropriate in terms of corroborating that the pursue of the greatest market share by means of decreasing markups can be disadvantageous

- Phase IV: While in all other experiments no differentiation was imposed to the Firms' probabilities of iterating with the Consumers, now the Companies were able to make adjustments in their advertising budgets with the objective of altering that feature. Two rounds of simulations were run, with four Firms playing. For the sake of comparison, in the first game the Companies had equivalent advertising expenses, contrasting with different ones in the second. The main finding in this phase was that increasing advertising expenses in the presence of more efficient competition

has demonstrated to be useless, because the Consumers rapidly turn their attention to the deals where they can find better advantage, measured by the obtained payoffs. In fact, due to the penalty payoff imposed to the sellers when a Consumer refuses to buy in an iteration, increasing the frequency of iterations can even be worse regarding the capacity of accumulating profits.

As could be perceived from the analysis of the simulations performed in Chapter 7, the model of the 1S-FIPD is practically unending in terms of possible experimentation. Many combinations of costs, prices, costs, number of Firms, range of the competition and advertising budgets can be implemented, apart from other lower-level adjustments in the parameters employed for the Consumers' decision. For this reason, the examples presented and discussed have been limited, and experiments focused on a selected number of situations, with the objective of demonstrating the potential of the process.

Finally, it is very important to emphasize here that the proposed method is intended as an analysis tool. Therefore, the parameters of the models, as well as the data employed in the simulations did *not* come from empirical observations, and have been arbitrarily but judiciously selected to mimic real data. As a consequence, the calibration of the model and its application to true sets of information remains as a suggestion for further improvement.

8.3 - Main Contributions of the Work

The scientific contributions presented by the present dissertation can be summarized by the following topics:

1. The admittance of gradual decisions in the Prisoner's Dilemma using elements of Fuzzy Set Theory;
2. The use of Belief Theory associated with Fuzzy Logic to implement decisions in a conflict of interest;
3. Introduction of adjustment factors in the method of the Fuzzy Integral;
4. Association of simulation methods with fuzzy decision criteria for the IPD;

5. Development of an analysis tool with an actual and practical application to the paradigm of the IPD;

8.4 - Limitations of the Work and Suggestions for Future Developments

a. Establishment of the general structures of the models of the FIPD

Besides fuzzy expert systems, other architectures could have been employed instead. For example, the techniques of genetic algorithms, multi-layered perceptrons (Neural Networks) and evolutionary programming (with Finite State Machines—FSMs) have already been investigated, as discussed in Chapter 3. But, by the time that this text is presented, the mentioned AI methods lack the use of fuzziness in their approach of the IPD. Therefore, those seem to be a promising line of research for building other realistic models of the IPD.

b. Choice of the variables

Selecting variables that can realistically mimic rational human behavior, represented by the agents' actions and perceptions in an IPD, is a challenging task. That problem is intimately correlated to finding an efficient frame to portray preferences, rationality and utility. In both models presented, a departure from the usually adopted information sets depicted by the “sequence of the opponent's last moves” has been sought and implemented. However, the current choice of variables on which the actors count to make their decisions provide just a possible configuration, and many other influencing factors could have been ascribed to the method. In this manner, other opportunities of research are open in that field, by alternatively picking diverse variables to mirror a rational decision process. Those may include psychological factors regarding the agents, environmental dynamic changes and influence of extraneous contingencies, among several others.

c. Detailing of the models

The encounters between the players in the two models of the FIPD have been guided by previously arbitrated particular probabilistic methods, which parameters

remain static along the simulation course. This fact entails the speculation for a quest of an adaptive random rule to govern the pairings, that might be continuously induced by the outputs of the game, thereby embodying a recurrent system. The inclusion of this and other alike features in the design of a simulated IPD must be carefully done, though. This suggestion regards the risk of ending up with a system which has some sort of chaotic dynamics embedded in it, which would be of rather complex analysis and could also lead to no further understanding of the nature of the conflict of interest depicted by the IPD.

An important aspect worth mentioning refers to the *tuning* of the fuzzy expert systems. The choice of the numeric parameters that particularize the fuzzy sets employed were discretionary, and some singling out had to be done in order to attain the computational simulation programs and runs. Nevertheless, it might be desirable to adjust the parameters of the fuzzy system to a set of empirical data, what could be done in accordance to some specific query.

The production rules utilized had single antecedents. This particular countenance has been deliberately picked in order to shun the trap of a combinatorial explosion of feasible individual strategies for playing the game. Recall that in the model of Chapter 5, a choice of three decision factors, each also with three qualitative assessments and only two strategies, namely, COOPERATE and DEFECT, led to 512 distinct fuzzy players. As a further development, it is proposed that the players may count on two-antecedent³ decision rules, which would be aggregated and mapped into the consequent through T-norms or S-norms fuzzy operators or still any other operator from a palette of options, including weighted and nonmonotonic fuzzy operators⁴.

³ Although multi-antecedent rules (with more than two) could be used, differently from control problems, it appears that it is not the case that the decisions adopted by human agents in a PD-like conflict of interest would take that course.

⁴ Regarding nonmonotonic aggregation operations, Yager and Filev (op. cit. [YAGE94], p. 40) state that, in order to do default or commonsense reasoning, nonmonotonic logic is more appropriate, which entails the utilization of that class of operators.

d. Relaxation of pairwise iteration - n-person IPD

Despite the diversity of individual types of players considered, the iterations were always pairwise. Other forms of pd-like situations involving three or more players (e. g. the *stag hunt*, the *free-rider* problem) might take advantage of a fuzzy approach to the decision method.

e. Allowance of features belonging to cooperative games

The PD is an essentially competitive game, though obviously non-strictly so. This is a necessary condition for it to possess the strikingly contradiction between the individual and the collective rationality. But it must be conceded that the restriction regarding the nonexistence of bargaining or negotiation between the parties participating in the dispute— exercised throughout the formulation of the presented method —is remarkably strong. On the other hand, total communication and enforcement of agreed contracts would completely dissolve the dilemma posed by the PD paradigm. Thus, under the same *rationale* that inspired the departure from the dichotomic game, a gradual compromise amid the contenders looks as though viable to be modeled and simulated in an eventual upcoming exploration of the theme.

f. Further Simulations

For both models of the FIPD, the simulations run with the respective programs cover only a part of the possibilities, that can be taken as illimitable. Hence, additional probing can always be realized, perhaps with a particular emphasis to some specific concerns whose clarification is sought.

8.4 - Final Remarks

Philosophers are intrigued by the Prisoner's Dilemma. The incongruity embedded in it is appalling because a widespread individual rationality leads not only to collective irrationality, but also to individual losses and, therefore, irrationality. The PD paradigm,

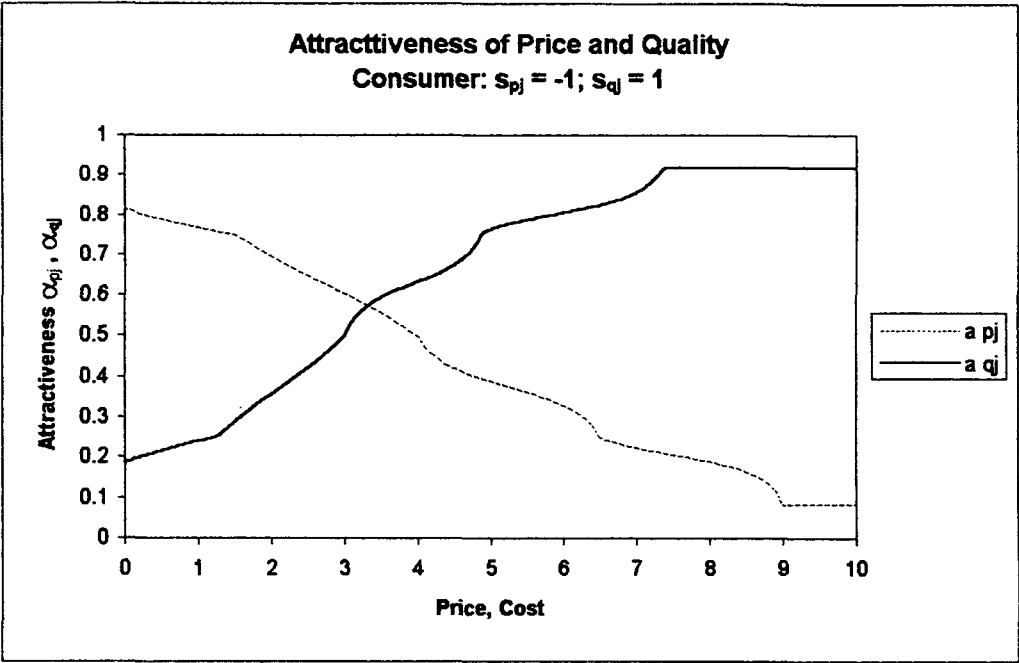
particularly its repeated version—IPD—, has been extensively employed to model conflicts of interest that turn up in a diversity of scientific areas, from operations research to social sciences and biology. The importance of the PD within the realm of Game Theory is unquestionable, since it reflects with both simplicity and clarity what this branch of knowledge is all about. On its turn, Game Theory is captivating an increasing interest as a powerful tool for developing application models aimed at real life situations and problems where rational and reactive actions are extant. A confirmation of the recognition of this trend was the granting of the 1994 Nobel Prize in Economics to three Game Theorists.

| | | | | | | | | | |
|-------|--------|-------|--------|-------|--------|-------|--------|-------|--------|
| 0.053 | 0.0757 | 0.253 | 0.2234 | 0.453 | 0.3510 | 0.653 | 0.4865 | 0.853 | 0.6585 |
| 0.054 | 0.0767 | 0.254 | 0.2241 | 0.454 | 0.3516 | 0.654 | 0.4872 | 0.854 | 0.6595 |
| 0.055 | 0.0776 | 0.255 | 0.2247 | 0.455 | 0.3523 | 0.655 | 0.4879 | 0.855 | 0.6606 |
| 0.056 | 0.0785 | 0.256 | 0.2254 | 0.456 | 0.3529 | 0.656 | 0.4887 | 0.856 | 0.6617 |
| 0.057 | 0.0795 | 0.257 | 0.2260 | 0.457 | 0.3535 | 0.657 | 0.4894 | 0.857 | 0.6628 |
| 0.058 | 0.0804 | 0.258 | 0.2267 | 0.458 | 0.3542 | 0.658 | 0.4901 | 0.858 | 0.6638 |
| 0.059 | 0.0813 | 0.259 | 0.2273 | 0.459 | 0.3548 | 0.659 | 0.4909 | 0.859 | 0.6649 |
| 0.06 | 0.0822 | 0.26 | 0.2280 | 0.46 | 0.3555 | 0.66 | 0.4916 | 0.86 | 0.6660 |
| 0.061 | 0.0832 | 0.261 | 0.2286 | 0.461 | 0.3561 | 0.661 | 0.4923 | 0.861 | 0.6671 |
| 0.062 | 0.0841 | 0.262 | 0.2292 | 0.462 | 0.3568 | 0.662 | 0.4931 | 0.862 | 0.6682 |
| 0.063 | 0.0850 | 0.263 | 0.2299 | 0.463 | 0.3574 | 0.663 | 0.4938 | 0.863 | 0.6693 |
| 0.064 | 0.0859 | 0.264 | 0.2305 | 0.464 | 0.3581 | 0.664 | 0.4946 | 0.864 | 0.6704 |
| 0.065 | 0.0868 | 0.265 | 0.2312 | 0.465 | 0.3587 | 0.665 | 0.4953 | 0.865 | 0.6716 |
| 0.066 | 0.0877 | 0.266 | 0.2318 | 0.466 | 0.3593 | 0.666 | 0.4960 | 0.866 | 0.6727 |
| 0.067 | 0.0886 | 0.267 | 0.2325 | 0.467 | 0.3600 | 0.667 | 0.4968 | 0.867 | 0.6738 |
| 0.068 | 0.0894 | 0.268 | 0.2331 | 0.468 | 0.3606 | 0.668 | 0.4975 | 0.868 | 0.6749 |
| 0.069 | 0.0903 | 0.269 | 0.2338 | 0.469 | 0.3613 | 0.669 | 0.4983 | 0.869 | 0.6761 |
| 0.07 | 0.0912 | 0.27 | 0.2344 | 0.47 | 0.3619 | 0.67 | 0.4990 | 0.87 | 0.6772 |
| 0.071 | 0.0921 | 0.271 | 0.2351 | 0.471 | 0.3626 | 0.671 | 0.4998 | 0.871 | 0.6783 |
| 0.072 | 0.0929 | 0.272 | 0.2357 | 0.472 | 0.3632 | 0.672 | 0.5005 | 0.872 | 0.6795 |
| 0.073 | 0.0938 | 0.273 | 0.2364 | 0.473 | 0.3639 | 0.673 | 0.5013 | 0.873 | 0.6806 |
| 0.074 | 0.0947 | 0.274 | 0.2370 | 0.474 | 0.3645 | 0.674 | 0.5020 | 0.874 | 0.6818 |
| 0.075 | 0.0955 | 0.275 | 0.2376 | 0.475 | 0.3652 | 0.675 | 0.5028 | 0.875 | 0.6829 |
| 0.076 | 0.0964 | 0.276 | 0.2383 | 0.476 | 0.3658 | 0.676 | 0.5035 | 0.876 | 0.6841 |
| 0.077 | 0.0972 | 0.277 | 0.2389 | 0.477 | 0.3664 | 0.677 | 0.5043 | 0.877 | 0.6853 |
| 0.078 | 0.0981 | 0.278 | 0.2396 | 0.478 | 0.3671 | 0.678 | 0.5050 | 0.878 | 0.6864 |
| 0.079 | 0.0989 | 0.279 | 0.2402 | 0.479 | 0.3677 | 0.679 | 0.5058 | 0.879 | 0.6876 |
| 0.08 | 0.0998 | 0.28 | 0.2409 | 0.48 | 0.3684 | 0.68 | 0.5065 | 0.88 | 0.6888 |
| 0.081 | 0.1006 | 0.281 | 0.2415 | 0.481 | 0.3690 | 0.681 | 0.5073 | 0.881 | 0.6900 |
| 0.082 | 0.1015 | 0.282 | 0.2422 | 0.482 | 0.3697 | 0.682 | 0.5080 | 0.882 | 0.6912 |
| 0.083 | 0.1023 | 0.283 | 0.2428 | 0.483 | 0.3703 | 0.683 | 0.5088 | 0.883 | 0.6924 |
| 0.084 | 0.1031 | 0.284 | 0.2434 | 0.484 | 0.3710 | 0.684 | 0.5096 | 0.884 | 0.6936 |
| 0.085 | 0.1040 | 0.285 | 0.2441 | 0.485 | 0.3716 | 0.685 | 0.5103 | 0.885 | 0.6948 |
| 0.086 | 0.1048 | 0.286 | 0.2447 | 0.486 | 0.3723 | 0.686 | 0.5111 | 0.886 | 0.6960 |
| 0.087 | 0.1056 | 0.287 | 0.2454 | 0.487 | 0.3729 | 0.687 | 0.5118 | 0.887 | 0.6972 |
| 0.088 | 0.1064 | 0.288 | 0.2460 | 0.488 | 0.3736 | 0.688 | 0.5126 | 0.888 | 0.6984 |
| 0.089 | 0.1072 | 0.289 | 0.2466 | 0.489 | 0.3742 | 0.689 | 0.5134 | 0.889 | 0.6996 |
| 0.09 | 0.1081 | 0.29 | 0.2473 | 0.49 | 0.3749 | 0.69 | 0.5141 | 0.89 | 0.7009 |
| 0.091 | 0.1089 | 0.291 | 0.2479 | 0.491 | 0.3755 | 0.691 | 0.5149 | 0.891 | 0.7021 |
| 0.092 | 0.1097 | 0.292 | 0.2486 | 0.492 | 0.3762 | 0.692 | 0.5157 | 0.892 | 0.7034 |
| 0.093 | 0.1105 | 0.293 | 0.2492 | 0.493 | 0.3768 | 0.693 | 0.5164 | 0.893 | 0.7046 |
| 0.094 | 0.1113 | 0.294 | 0.2498 | 0.494 | 0.3775 | 0.694 | 0.5172 | 0.894 | 0.7059 |
| 0.095 | 0.1121 | 0.295 | 0.2505 | 0.495 | 0.3781 | 0.695 | 0.5180 | 0.895 | 0.7071 |
| 0.096 | 0.1129 | 0.296 | 0.2511 | 0.496 | 0.3788 | 0.696 | 0.5187 | 0.896 | 0.7084 |
| 0.097 | 0.1137 | 0.297 | 0.2518 | 0.497 | 0.3794 | 0.697 | 0.5195 | 0.897 | 0.7097 |
| 0.098 | 0.1145 | 0.298 | 0.2524 | 0.498 | 0.3801 | 0.698 | 0.5203 | 0.898 | 0.7110 |
| 0.099 | 0.1153 | 0.299 | 0.2530 | 0.499 | 0.3808 | 0.699 | 0.5211 | 0.899 | 0.7122 |
| 0.1 | 0.1161 | 0.3 | 0.2537 | 0.5 | 0.3814 | 0.7 | 0.5218 | 0.9 | 0.7135 |
| 0.101 | 0.1169 | 0.301 | 0.2543 | 0.501 | 0.3821 | 0.701 | 0.5226 | 0.901 | 0.7148 |
| 0.102 | 0.1177 | 0.302 | 0.2550 | 0.502 | 0.3827 | 0.702 | 0.5234 | 0.902 | 0.7161 |
| 0.103 | 0.1185 | 0.303 | 0.2556 | 0.503 | 0.3834 | 0.703 | 0.5242 | 0.903 | 0.7175 |
| 0.104 | 0.1192 | 0.304 | 0.2562 | 0.504 | 0.3840 | 0.704 | 0.5249 | 0.904 | 0.7188 |
| 0.105 | 0.1200 | 0.305 | 0.2569 | 0.505 | 0.3847 | 0.705 | 0.5257 | 0.905 | 0.7201 |

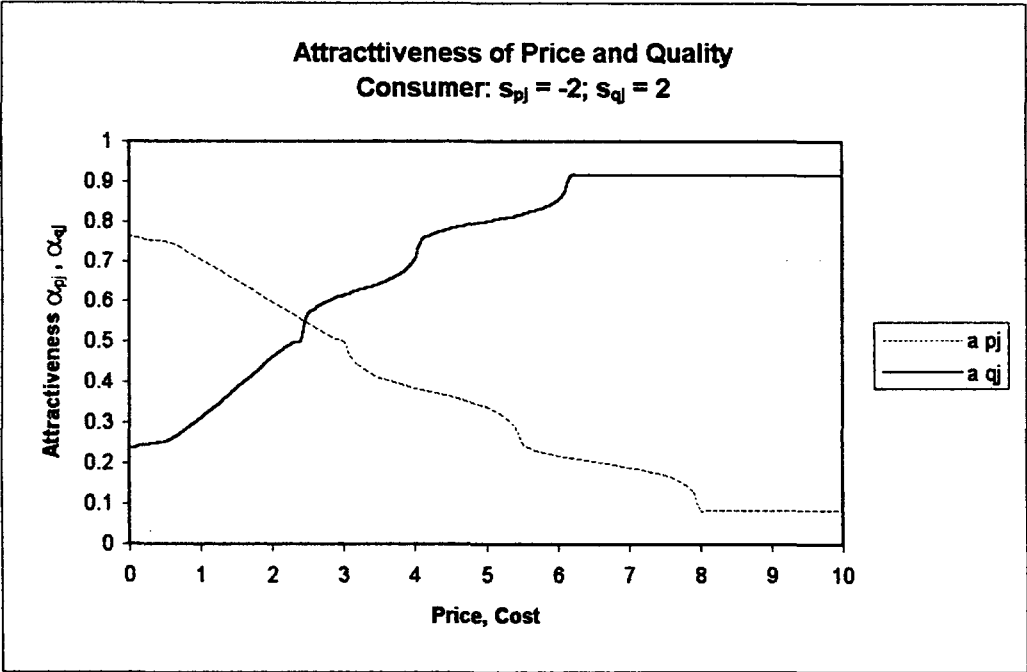
| | | | | | | | | | |
|-------|--------|-------|--------|-------|--------|-------|--------|-------|--------|
| 0.106 | 0.1208 | 0.306 | 0.2575 | 0.506 | 0.3853 | 0.706 | 0.5265 | 0.906 | 0.7214 |
| 0.107 | 0.1216 | 0.307 | 0.2582 | 0.507 | 0.3860 | 0.707 | 0.5273 | 0.907 | 0.7228 |
| 0.108 | 0.1224 | 0.308 | 0.2588 | 0.508 | 0.3866 | 0.708 | 0.5281 | 0.908 | 0.7241 |
| 0.109 | 0.1231 | 0.309 | 0.2594 | 0.509 | 0.3873 | 0.709 | 0.5288 | 0.909 | 0.7255 |
| 0.11 | 0.1239 | 0.31 | 0.2601 | 0.51 | 0.3880 | 0.71 | 0.5296 | 0.91 | 0.7269 |
| 0.111 | 0.1247 | 0.311 | 0.2607 | 0.511 | 0.3886 | 0.711 | 0.5304 | 0.911 | 0.7282 |
| 0.112 | 0.1255 | 0.312 | 0.2613 | 0.512 | 0.3893 | 0.712 | 0.5312 | 0.912 | 0.7296 |
| 0.113 | 0.1262 | 0.313 | 0.2620 | 0.513 | 0.3899 | 0.713 | 0.5320 | 0.913 | 0.7310 |
| 0.114 | 0.1270 | 0.314 | 0.2626 | 0.514 | 0.3906 | 0.714 | 0.5328 | 0.914 | 0.7324 |
| 0.115 | 0.1278 | 0.315 | 0.2633 | 0.515 | 0.3912 | 0.715 | 0.5336 | 0.915 | 0.7338 |
| 0.116 | 0.1285 | 0.316 | 0.2639 | 0.516 | 0.3919 | 0.716 | 0.5344 | 0.916 | 0.7352 |
| 0.117 | 0.1293 | 0.317 | 0.2645 | 0.517 | 0.3926 | 0.717 | 0.5352 | 0.917 | 0.7366 |
| 0.118 | 0.1300 | 0.318 | 0.2652 | 0.518 | 0.3932 | 0.718 | 0.5359 | 0.918 | 0.7381 |
| 0.119 | 0.1308 | 0.319 | 0.2658 | 0.519 | 0.3939 | 0.719 | 0.5367 | 0.919 | 0.7395 |
| 0.12 | 0.1316 | 0.32 | 0.2664 | 0.52 | 0.3945 | 0.72 | 0.5375 | 0.92 | 0.7410 |
| 0.121 | 0.1323 | 0.321 | 0.2671 | 0.521 | 0.3952 | 0.721 | 0.5383 | 0.921 | 0.7424 |
| 0.122 | 0.1331 | 0.322 | 0.2677 | 0.522 | 0.3959 | 0.722 | 0.5391 | 0.922 | 0.7439 |
| 0.123 | 0.1338 | 0.323 | 0.2683 | 0.523 | 0.3965 | 0.723 | 0.5399 | 0.923 | 0.7454 |
| 0.124 | 0.1346 | 0.324 | 0.2690 | 0.524 | 0.3972 | 0.724 | 0.5407 | 0.924 | 0.7469 |
| 0.125 | 0.1353 | 0.325 | 0.2696 | 0.525 | 0.3978 | 0.725 | 0.5415 | 0.925 | 0.7484 |
| 0.126 | 0.1361 | 0.326 | 0.2703 | 0.526 | 0.3985 | 0.726 | 0.5423 | 0.926 | 0.7499 |
| 0.127 | 0.1368 | 0.327 | 0.2709 | 0.527 | 0.3992 | 0.727 | 0.5432 | 0.927 | 0.7514 |
| 0.128 | 0.1376 | 0.328 | 0.2715 | 0.528 | 0.3998 | 0.728 | 0.5440 | 0.928 | 0.7529 |
| 0.129 | 0.1383 | 0.329 | 0.2722 | 0.529 | 0.4005 | 0.729 | 0.5448 | 0.929 | 0.7545 |
| 0.13 | 0.1390 | 0.33 | 0.2728 | 0.53 | 0.4012 | 0.73 | 0.5456 | 0.93 | 0.7560 |
| 0.131 | 0.1398 | 0.331 | 0.2734 | 0.531 | 0.4018 | 0.731 | 0.5464 | 0.931 | 0.7576 |
| 0.132 | 0.1405 | 0.332 | 0.2741 | 0.532 | 0.4025 | 0.732 | 0.5472 | 0.932 | 0.7592 |
| 0.133 | 0.1413 | 0.333 | 0.2747 | 0.533 | 0.4031 | 0.733 | 0.5480 | 0.933 | 0.7608 |
| 0.134 | 0.1420 | 0.334 | 0.2753 | 0.534 | 0.4038 | 0.734 | 0.5488 | 0.934 | 0.7624 |
| 0.135 | 0.1427 | 0.335 | 0.2760 | 0.535 | 0.4045 | 0.735 | 0.5496 | 0.935 | 0.7640 |
| 0.136 | 0.1435 | 0.336 | 0.2766 | 0.536 | 0.4051 | 0.736 | 0.5505 | 0.936 | 0.7656 |
| 0.137 | 0.1442 | 0.337 | 0.2772 | 0.537 | 0.4058 | 0.737 | 0.5513 | 0.937 | 0.7672 |
| 0.138 | 0.1449 | 0.338 | 0.2779 | 0.538 | 0.4065 | 0.738 | 0.5521 | 0.938 | 0.7689 |
| 0.139 | 0.1457 | 0.339 | 0.2785 | 0.539 | 0.4071 | 0.739 | 0.5529 | 0.939 | 0.7706 |
| 0.14 | 0.1464 | 0.34 | 0.2791 | 0.54 | 0.4078 | 0.74 | 0.5537 | 0.94 | 0.7723 |
| 0.141 | 0.1471 | 0.341 | 0.2798 | 0.541 | 0.4085 | 0.741 | 0.5546 | 0.941 | 0.7740 |
| 0.142 | 0.1478 | 0.342 | 0.2804 | 0.542 | 0.4091 | 0.742 | 0.5554 | 0.942 | 0.7757 |
| 0.143 | 0.1486 | 0.343 | 0.2810 | 0.543 | 0.4098 | 0.743 | 0.5562 | 0.943 | 0.7774 |
| 0.144 | 0.1493 | 0.344 | 0.2817 | 0.544 | 0.4105 | 0.744 | 0.5570 | 0.944 | 0.7791 |
| 0.145 | 0.1500 | 0.345 | 0.2823 | 0.545 | 0.4111 | 0.745 | 0.5579 | 0.945 | 0.7809 |
| 0.146 | 0.1507 | 0.346 | 0.2829 | 0.546 | 0.4118 | 0.746 | 0.5587 | 0.946 | 0.7827 |
| 0.147 | 0.1515 | 0.347 | 0.2836 | 0.547 | 0.4125 | 0.747 | 0.5595 | 0.947 | 0.7845 |
| 0.148 | 0.1522 | 0.348 | 0.2842 | 0.548 | 0.4132 | 0.748 | 0.5604 | 0.948 | 0.7863 |
| 0.149 | 0.1529 | 0.349 | 0.2848 | 0.549 | 0.4138 | 0.749 | 0.5612 | 0.949 | 0.7881 |
| 0.15 | 0.1536 | 0.35 | 0.2855 | 0.55 | 0.4145 | 0.75 | 0.5620 | 0.95 | 0.7900 |
| 0.151 | 0.1543 | 0.351 | 0.2861 | 0.551 | 0.4152 | 0.751 | 0.5629 | 0.951 | 0.7919 |
| 0.152 | 0.1550 | 0.352 | 0.2867 | 0.552 | 0.4158 | 0.752 | 0.5637 | 0.952 | 0.7938 |
| 0.153 | 0.1558 | 0.353 | 0.2874 | 0.553 | 0.4165 | 0.753 | 0.5645 | 0.953 | 0.7957 |
| 0.154 | 0.1565 | 0.354 | 0.2880 | 0.554 | 0.4172 | 0.754 | 0.5654 | 0.954 | 0.7976 |
| 0.155 | 0.1572 | 0.355 | 0.2886 | 0.555 | 0.4179 | 0.755 | 0.5662 | 0.955 | 0.7996 |
| 0.156 | 0.1579 | 0.356 | 0.2893 | 0.556 | 0.4185 | 0.756 | 0.5671 | 0.956 | 0.8016 |
| 0.157 | 0.1586 | 0.357 | 0.2899 | 0.557 | 0.4192 | 0.757 | 0.5679 | 0.957 | 0.8036 |
| 0.158 | 0.1593 | 0.358 | 0.2905 | 0.558 | 0.4199 | 0.758 | 0.5688 | 0.958 | 0.8056 |

| | | | | | | | | | |
|-------|--------|-------|--------|-------|--------|-------|--------|--------|--------|
| 0.159 | 0.1600 | 0.359 | 0.2912 | 0.559 | 0.4206 | 0.759 | 0.5696 | 0.959 | 0.8077 |
| 0.16 | 0.1607 | 0.36 | 0.2918 | 0.56 | 0.4212 | 0.76 | 0.5705 | 0.96 | 0.8097 |
| 0.161 | 0.1614 | 0.361 | 0.2925 | 0.561 | 0.4219 | 0.761 | 0.5713 | 0.961 | 0.8119 |
| 0.162 | 0.1621 | 0.362 | 0.2931 | 0.562 | 0.4226 | 0.762 | 0.5722 | 0.962 | 0.8140 |
| 0.163 | 0.1628 | 0.363 | 0.2937 | 0.563 | 0.4233 | 0.763 | 0.5730 | 0.963 | 0.8162 |
| 0.164 | 0.1635 | 0.364 | 0.2944 | 0.564 | 0.4239 | 0.764 | 0.5739 | 0.964 | 0.8184 |
| 0.165 | 0.1642 | 0.365 | 0.2950 | 0.565 | 0.4246 | 0.765 | 0.5747 | 0.965 | 0.8206 |
| 0.166 | 0.1649 | 0.366 | 0.2956 | 0.566 | 0.4253 | 0.766 | 0.5756 | 0.966 | 0.8229 |
| 0.167 | 0.1656 | 0.367 | 0.2963 | 0.567 | 0.4260 | 0.767 | 0.5765 | 0.967 | 0.8252 |
| 0.168 | 0.1663 | 0.368 | 0.2969 | 0.568 | 0.4267 | 0.768 | 0.5773 | 0.968 | 0.8276 |
| 0.169 | 0.1670 | 0.369 | 0.2975 | 0.569 | 0.4273 | 0.769 | 0.5782 | 0.969 | 0.8299 |
| 0.17 | 0.1677 | 0.37 | 0.2982 | 0.57 | 0.4280 | 0.77 | 0.5791 | 0.97 | 0.8324 |
| 0.171 | 0.1684 | 0.371 | 0.2988 | 0.571 | 0.4287 | 0.771 | 0.5799 | 0.971 | 0.8349 |
| 0.172 | 0.1691 | 0.372 | 0.2994 | 0.572 | 0.4294 | 0.772 | 0.5808 | 0.972 | 0.8374 |
| 0.173 | 0.1698 | 0.373 | 0.3001 | 0.573 | 0.4301 | 0.773 | 0.5817 | 0.973 | 0.8400 |
| 0.174 | 0.1705 | 0.374 | 0.3007 | 0.574 | 0.4307 | 0.774 | 0.5825 | 0.974 | 0.8426 |
| 0.175 | 0.1712 | 0.375 | 0.3013 | 0.575 | 0.4314 | 0.775 | 0.5834 | 0.975 | 0.8453 |
| 0.176 | 0.1719 | 0.376 | 0.3020 | 0.576 | 0.4321 | 0.776 | 0.5843 | 0.976 | 0.8480 |
| 0.177 | 0.1726 | 0.377 | 0.3026 | 0.577 | 0.4328 | 0.777 | 0.5852 | 0.977 | 0.8508 |
| 0.178 | 0.1733 | 0.378 | 0.3032 | 0.578 | 0.4335 | 0.778 | 0.5861 | 0.978 | 0.8537 |
| 0.179 | 0.1740 | 0.379 | 0.3039 | 0.579 | 0.4342 | 0.779 | 0.5869 | 0.979 | 0.8566 |
| 0.18 | 0.1747 | 0.38 | 0.3045 | 0.58 | 0.4348 | 0.78 | 0.5878 | 0.98 | 0.8596 |
| 0.181 | 0.1754 | 0.381 | 0.3051 | 0.581 | 0.4355 | 0.781 | 0.5887 | 0.981 | 0.8628 |
| 0.182 | 0.1761 | 0.382 | 0.3058 | 0.582 | 0.4362 | 0.782 | 0.5896 | 0.982 | 0.8660 |
| 0.183 | 0.1767 | 0.383 | 0.3064 | 0.583 | 0.4369 | 0.783 | 0.5905 | 0.983 | 0.8693 |
| 0.184 | 0.1774 | 0.384 | 0.3070 | 0.584 | 0.4376 | 0.784 | 0.5914 | 0.984 | 0.8727 |
| 0.185 | 0.1781 | 0.385 | 0.3077 | 0.585 | 0.4383 | 0.785 | 0.5923 | 0.985 | 0.8762 |
| 0.186 | 0.1788 | 0.386 | 0.3083 | 0.586 | 0.4390 | 0.786 | 0.5932 | 0.986 | 0.8798 |
| 0.187 | 0.1795 | 0.387 | 0.3089 | 0.587 | 0.4397 | 0.787 | 0.5941 | 0.987 | 0.8837 |
| 0.188 | 0.1802 | 0.388 | 0.3096 | 0.588 | 0.4403 | 0.788 | 0.5950 | 0.988 | 0.8876 |
| 0.189 | 0.1808 | 0.389 | 0.3102 | 0.589 | 0.4410 | 0.789 | 0.5959 | 0.989 | 0.8918 |
| 0.19 | 0.1815 | 0.39 | 0.3108 | 0.59 | 0.4417 | 0.79 | 0.5968 | 0.99 | 0.8962 |
| 0.191 | 0.1822 | 0.391 | 0.3115 | 0.591 | 0.4424 | 0.791 | 0.5977 | 0.991 | 0.9008 |
| 0.192 | 0.1829 | 0.392 | 0.3121 | 0.592 | 0.4431 | 0.792 | 0.5986 | 0.992 | 0.9057 |
| 0.193 | 0.1836 | 0.393 | 0.3127 | 0.593 | 0.4438 | 0.793 | 0.5995 | 0.993 | 0.9110 |
| 0.194 | 0.1843 | 0.394 | 0.3134 | 0.594 | 0.4445 | 0.794 | 0.6004 | 0.994 | 0.9167 |
| 0.195 | 0.1849 | 0.395 | 0.3140 | 0.595 | 0.4452 | 0.795 | 0.6013 | 0.995 | 0.9230 |
| 0.196 | 0.1856 | 0.396 | 0.3146 | 0.596 | 0.4459 | 0.796 | 0.6022 | 0.996 | 0.9301 |
| 0.197 | 0.1863 | 0.397 | 0.3153 | 0.597 | 0.4466 | 0.797 | 0.6031 | 0.997 | 0.9382 |
| 0.198 | 0.1870 | 0.398 | 0.3159 | 0.598 | 0.4473 | 0.798 | 0.6041 | 0.998 | 0.9481 |
| 0.199 | 0.1876 | 0.399 | 0.3165 | 0.599 | 0.4480 | 0.799 | 0.6050 | 0.999 | 0.9614 |
| 0.2 | 0.1883 | 0.4 | 0.3172 | 0.6 | 0.4487 | 0.8 | 0.6059 | 0.9999 | 0.9856 |

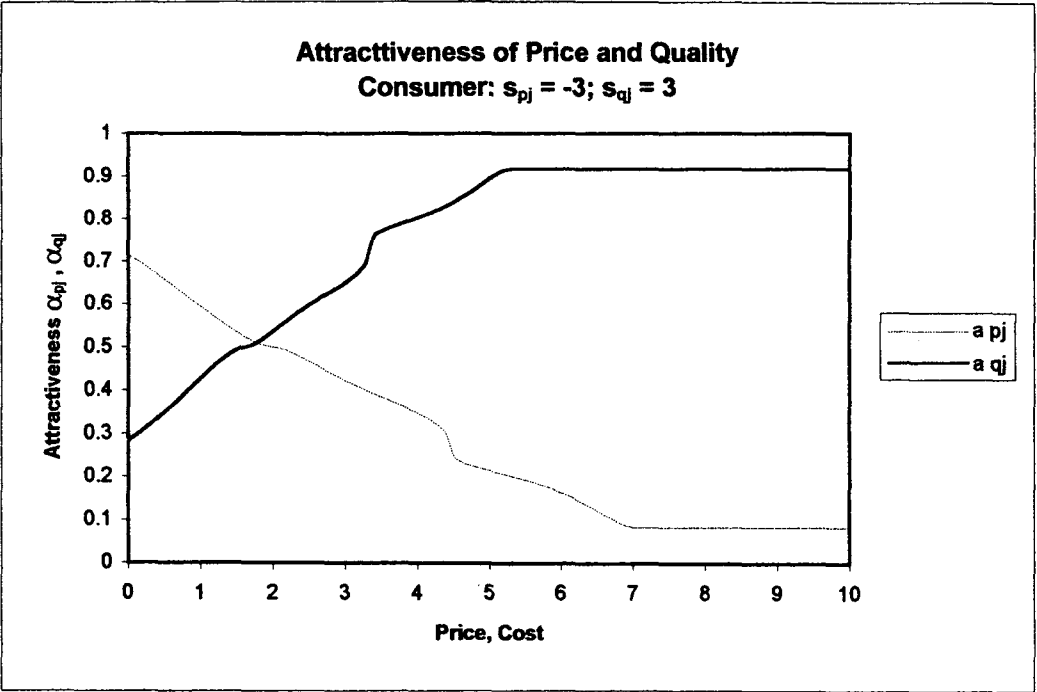
Attractiveness to Price(α_{pj}), Quality(α_{qj}) - Consumer with $s_{pj} = -1$



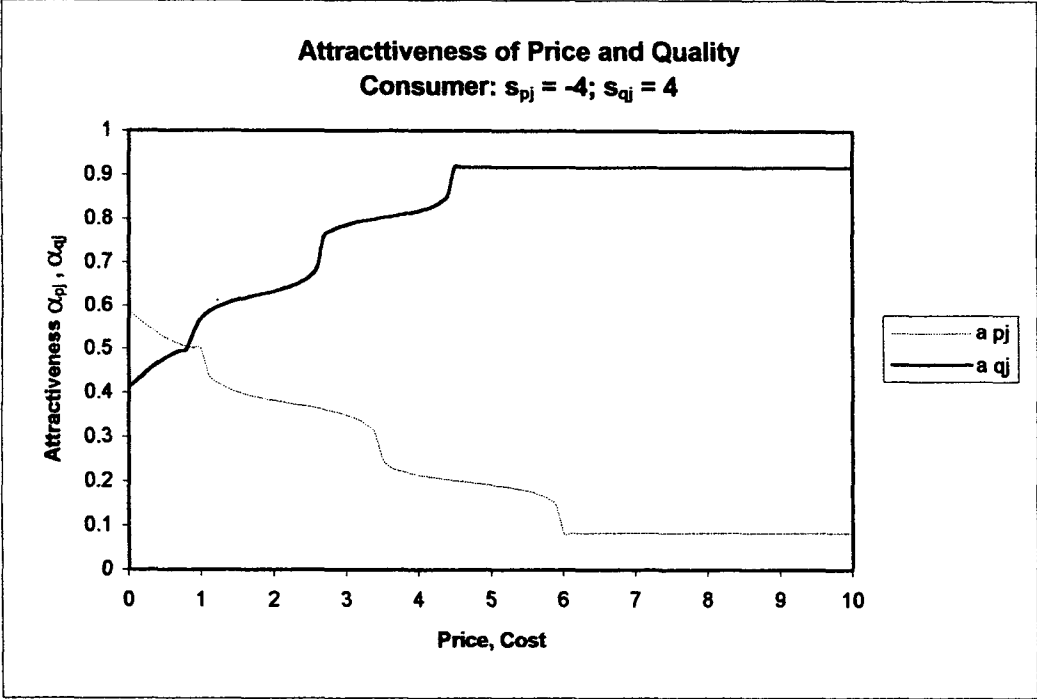
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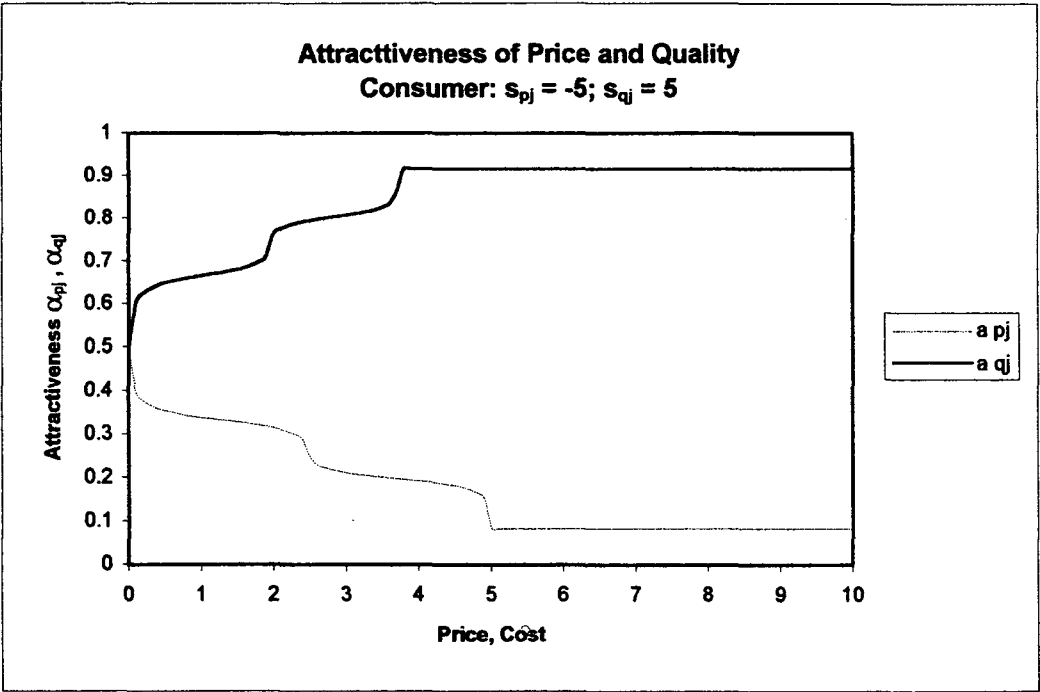
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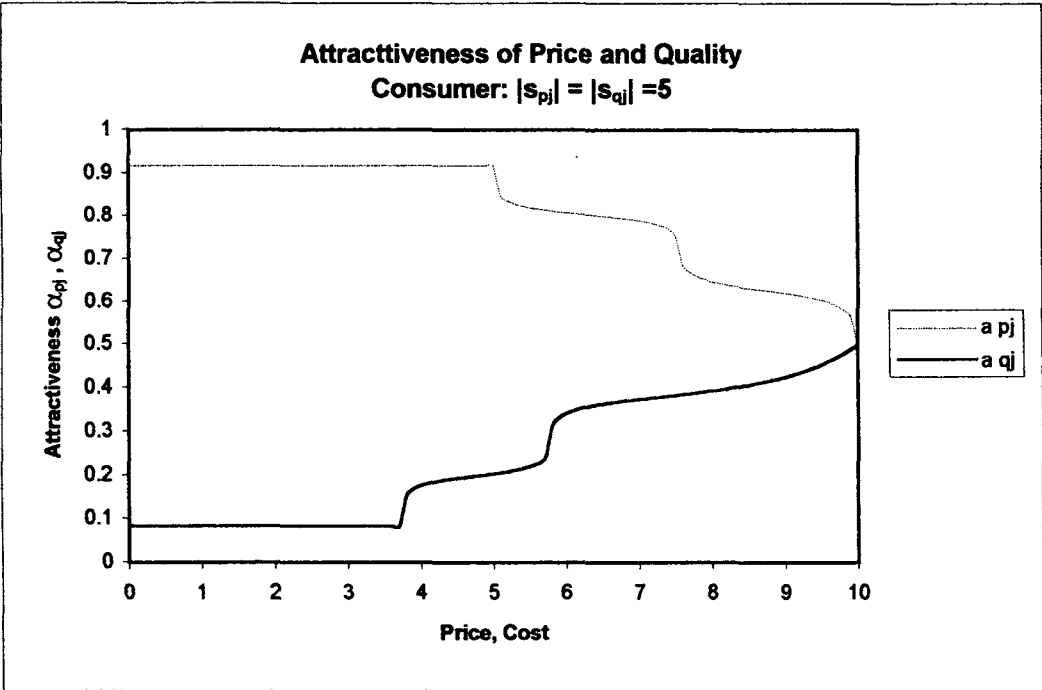
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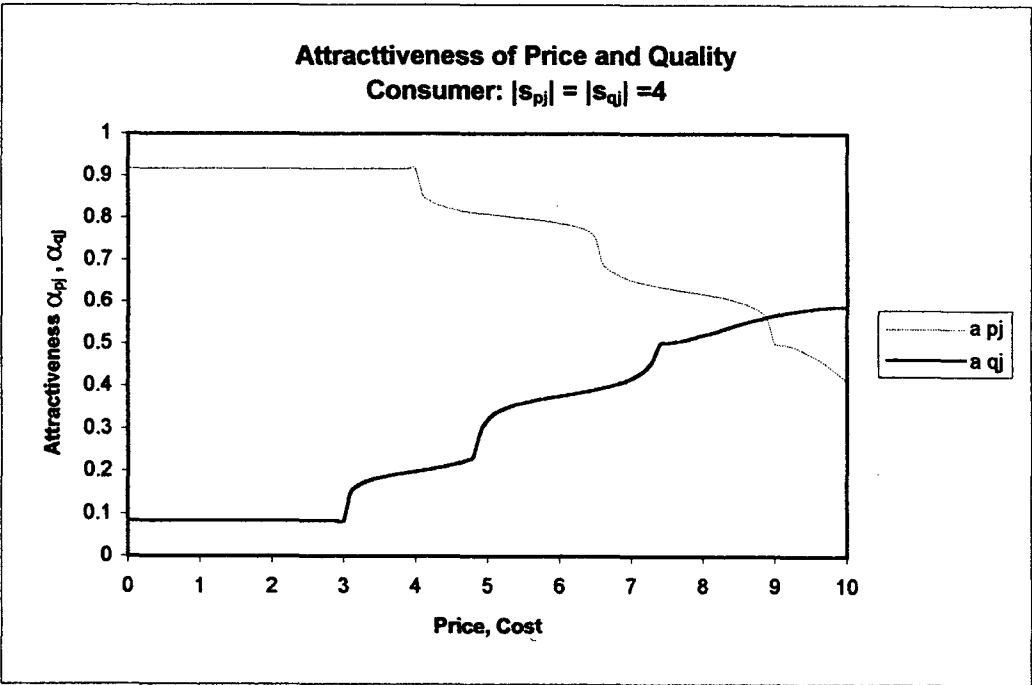
Attractiveness to Price(a_{pj}), Quality(a_{qj}) - Consumer with $s_{pj} = -5$



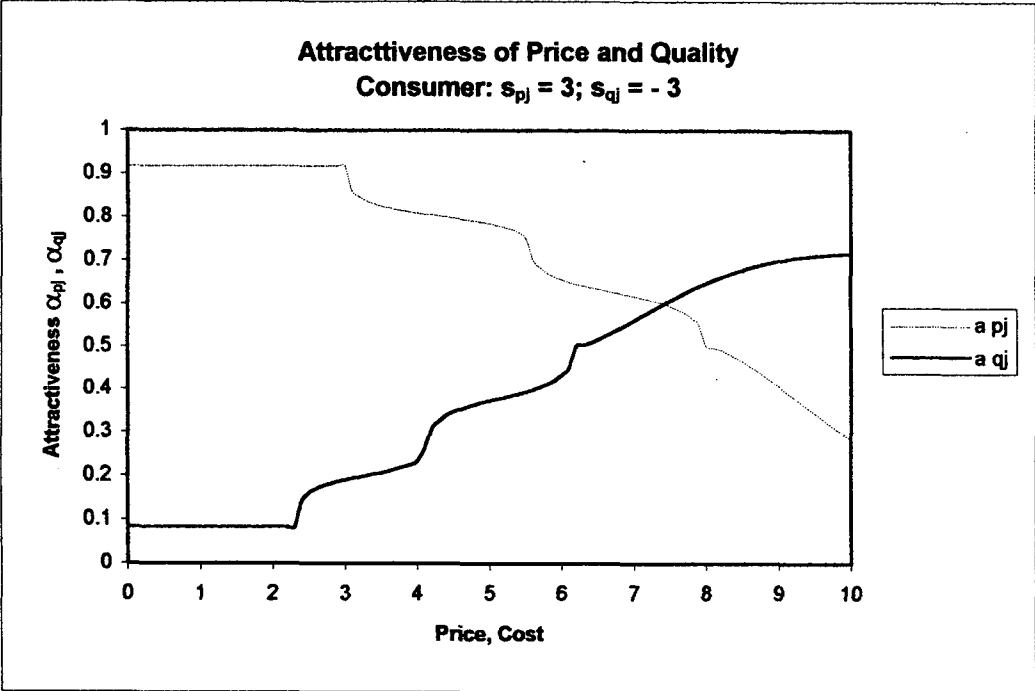
Attractiveness to Price(a_{pj}), Quality(a_{qj}) - Consumer with $s_{pj} = 5$



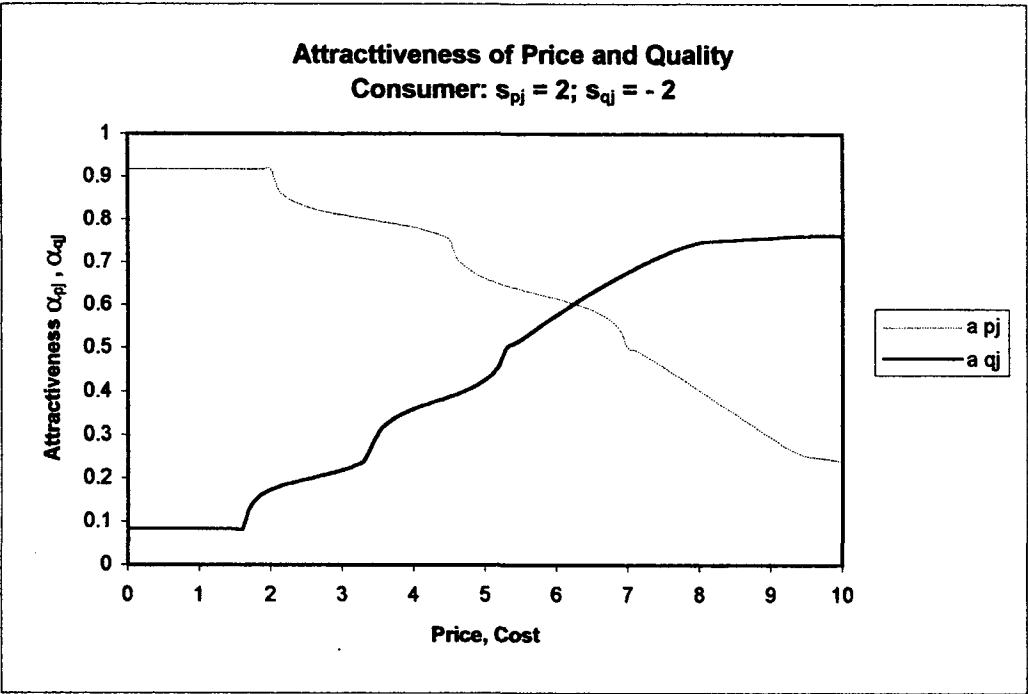
Attractiveness to Price(a_{pj}), Quality(a_{qj}) - Consumer with $s_{pj} = 4$



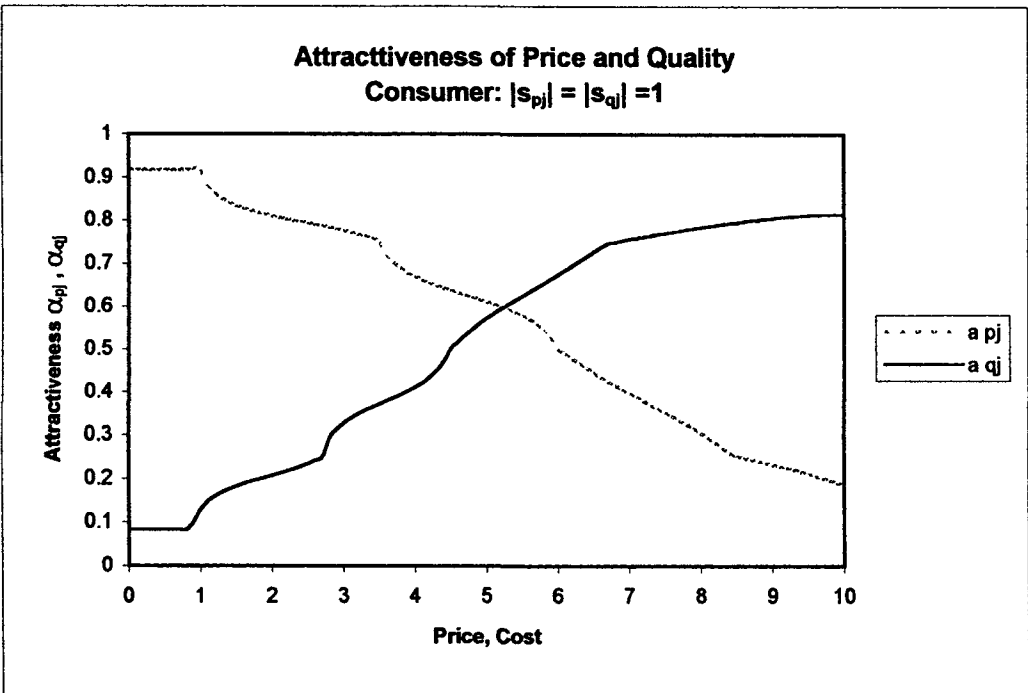
Attractiveness to Price(a_{pj}), Quality(a_{qj}) - Consumer with $s_{pj} = 3$



Attractiveness to Price(a_{pj}), Quality(a_{qj}) - Consumer with $s_{pj} = 2$



Attractiveness to Price(a_{pj}), Quality(a_{qj}) - Consumer with $s_{pj} = 1$



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